NPS ARCHIVE
1959
GUNNING, T.

MAGNETIC NOZZLING OF A PLASMA FLOW

THOMAS I. GUNNING

LIBRARY
U.S. NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

254

MAGNETIC NOZZLING
OF A PLASMA FLOW

* * * * *

Thomas I. Gunning

MAGNETIC NOZZLING
OF A PLASMA FLOW

by

Thomas I. Gunning

Major, United States Marine Corps

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1 9 5 9

NPS ARCHIVE
1959
GUNNING, T.

Thesis

MAGNETIC NOZZLING
OF A PLASMA FLOW

by

Thomas I. Gunning

This work is accepted as fulfilling
the thesis requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

from the

United States Naval Postgraduate School

ABSTRACT

The use of a magnetic field to contain plasma at extremely high temperatures has come to the fore in recent years as the temperatures in propulsion systems have risen. This thesis makes a theoretical investigation of the shape of a nozzle formed by a magnetic field acting on a perfectly conducting plasma. A description of the system with basic assumptions is given, and the system is treated analytically in an attempt to develop some method of deriving a closed form solution. A numerical attack on the problem is used to find actual nozzle shapes for a variety of conditions of Mach number, specific heat ratio, current, and distance between the current-carrying conductor and channel wall.

The author wishes to express his appreciation and gratitude to Professor Orval H. Polk of the United States Postgraduate School for the assistance and encouragement given him in this study. He also thanks Dr. Richard M. Head of the same school for arousing his interest in the field of magnetohydrodynamics and Dr. Burton D. Fried of Space Technology Laboratory, Hawthorne, California, whose work provided a basis for this investigation, and whose discussion with the author pointed out the problem areas needing investigation.

TABLE OF CONTENTS

Section	Title	Page
1.	Introduction	1
2.	System Description and Basic Mathematical Discussion	2
3.	Simplifying Assumptions	5
4.	Consequences of These Assumptions	6
5.	The Common Surface and its Imposed Boundary Conditions	8
6.	Current Criticality and its Effect on Nozzle Shape	13
7.	Discussion of a Mathematical Solution	15
8.	Numerical Solutions for Specific Cases	21
9.	Conclusions	31
10.	Bibliography	59

LIST OF ILLUSTRATIONS

Figure		Page
1.	Diagram of Basic System	3
2.	Variation of Nozzle Shape with Change in Current	35
3.	Variation of Nozzle Shape with Change in Mach Number	36
4.	Variation of Nozzle Shape with Change in Specific Heat Ratio	37
5.	Variation of Nozzle Shape with Change in Conductor to Wall Distance, Absolute Current Constant	38
6.	Variation of Nozzle Shape with Change in Conductor to Wall Distance, Number of Critical Currents Constant	39

TABLE OF SYMBOLS

a	Sonic velocity
A	Area of nozzle in equation 14; magnetic potential
B	Magnetic induction
c	Speed of light
c_v	Specific heat at constant volume
d	Distance from current-carrying conductor to point of given magnetic flux density
d_m	Distance from current-carrying conductor to nozzle breakaway point
D	Displacement density
$\frac{D}{Dt}$	Vector operator; $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$
E	Electric field intensity
f	y-coordinate of nozzle points
g	Function defined by equation 46
G	Green's function
h	Function defined by equation 49
H	Magnetic field strength
I	Electric current
I_c	Critical current value
j	Current density
l	Nozzle length
m	Abscissa of nozzle breakaway point
M	Mach number
n	Number of critical currents; unit normal vector
p	Fluid hydrodynamic pressure

TABLE OF SYMBOLS
continued

p_f	Fluid hydrodynamic pressure in Section 8
p_m	Pressure exerted by magnetic field
p_o	Free stream hydrodynamic pressure
P	Pressure function defined by equation 49
q	Charge density
r	Perpendicular distance between current-carrying conductor and channel wall
R	Universal gas constant
s	Arc length
S	Entropy
t	Time
T	Temperature
v	Velocity; specific volume in equation 9
y_o	y-coordinate of fixed portion of channel wall
γ	Specific heat ratio
δ	Dirac delta function; variational symbol
ϵ	Dielectric constant
λ	Scalar function defined in equation 11
μ	Magnetic permeability
ρ	Fluid density
σ	Fluid conductivity
τ	Viscous stress tensor
ω	Nozzle shaping factor

1. Introduction.

Fluid flow through some constraint such as a nozzle is a phenomenon which is rather commonplace in these days of rockets, jet propulsion, and high Mach number wind tunnels. Everyone is familiar with the use of a physical constraint in these applications. This is generally satisfactory as long as the conditions involved in the flow do not exceed the thermal and mechanical limitations imposed by the nature of the material used.

However, when the search for improved performances in these devices requires fluid temperatures which preclude the use of any known material, two approaches may be employed. New materials may be developed which operate satisfactorily under these conditions. While this may solve the problem temporarily, there still exists the possibility that the next step forward may create the same problem again under conditions making it even more difficult of solution. The other approach is to find some constraint which is not affected by temperatures. This could have application not only in the situation as outlined above, but in any set of circumstances where exceedingly high temperatures are encountered.

Thus, much interest is being shown in the containment of high temperature matter by magnetic fields. When the fluid employed is in a state of ionization, that is, exists as a plasma, these techniques are especially applicable. The problem of nozzling a plasma flow by means of a magnetic field is discussed in this thesis.

This thesis is primarily a theoretical discussion of the problem. It is recognized that the conditions established would probably not be those of a practical situation. However, it is felt that the discussion

could, by extension, be applied to other situations which would be more closely akin to actuality.

The thesis attempts to develop a mathematical basis for such a problem, using some assumptions which greatly simplify the mathematics while preserving the validity of the problem. A discussion of the analytic solution of the equations is made, which under proper conditions could possibly furnish at least a starting point for getting answers. Finally, a numerical solution of the problem, using arbitrarily chosen parameters is discussed. The results of these parameters are tabulated in detail and graphed in somewhat less detail to show qualitatively the effect on the system of parametric variations. These are then discussed and some general conclusions drawn. CGS units are used in all equations.

2. System Description and Basic Mathematical Discussion.

Consider a neutral, ionized fluid confined in a two-dimensional channel of width $2y_o$ as shown in Fig. 1. An infinitely long current-carrying conductor is located at a distance, r, above the upper wall of the channel, and oriented perpendicular to the direction of flow. It carries a steady current, I, in one direction. A similar conductor with current in the opposite direction is located identically beneath the lower channel wall.

The origin of a rectangular coordinate system is chosen at the intersection of the midplane of the channel and the line joining the two conductors, with x taken as positive to the right and y positive upward. Thus the channel mid-plane becomes the xz-coordinate plane. Because of the symmetry of the system about the x-axis, the shape of the upper half of the nozzle is the mirror image of the lower. Thus,

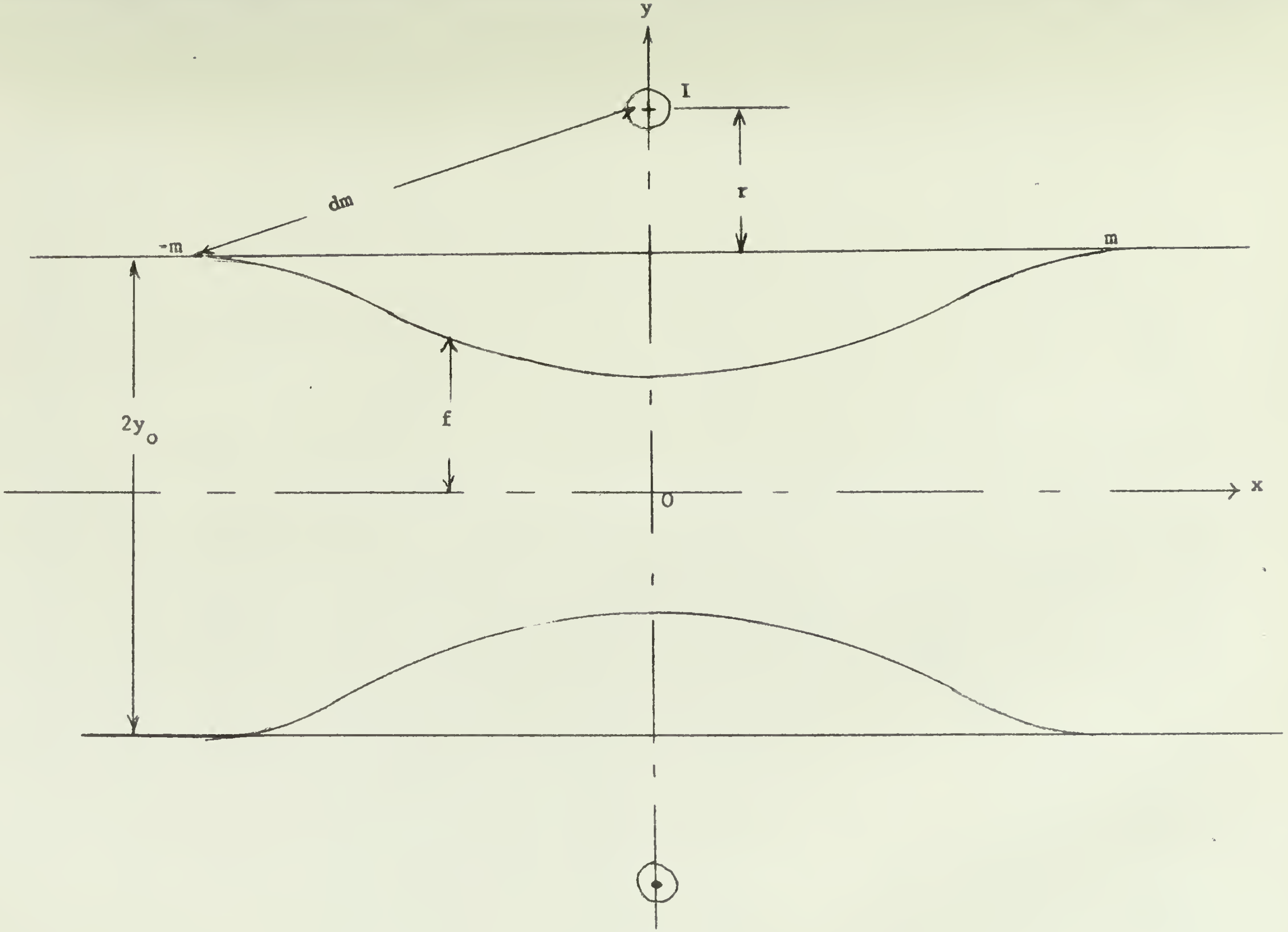


Fig. 1

Diagram of the Basic System

a solution for one will serve for both. The upper half is considered here.

In order to mathematically describe such a system[1,2,3] consider separately the phenomena affecting it. The electromagnetic aspect is at once apparent. The field effects from this source are best described by Maxwell's equations. These start with Ampere's law relating magnetic induction to the electric current producing it:

$$\nabla \times \vec{B} = \frac{4\pi \vec{j} - \frac{\partial \vec{E}}{\partial t}}{c} . \tag{1}$$

Second is Faraday's Law of Induction:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} . \tag{2}$$

The last of these equations concern the field-source relationships:

$$\nabla \cdot \vec{B} = 0, \tag{3}$$

$$\nabla \cdot \vec{E} = 4\pi q . \tag{4}$$

For the electric properties of the moving medium, Ohm's law is needed:

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right). \tag{5}$$

In considering the flowing fluid of the system, the normal conservation equations of hydrodynamics must still apply, modified now to include the electromagnetic action upon the flow. The equation for conservation of mass is unaffected:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{6}$$

[1] Alfvén, Cosmical Electrodynamics, Clarendon Press, Oxford, 1950

[2] Meyer, On Reducing Aerodynamic Heat Transfer Rates by Magnetohydrodynamic Techniques, IAS Preprint No. 816, 1958

[3] Clauser, Magnetohydrodynamics, STL Rpt PRL-8-04, 1958

The Lorentz force and the electric field force must be included in the momentum equation:

$$\rho \frac{D\vec{v}}{Dt} + \nabla p = q\vec{E} + \vec{j} \times \frac{\vec{B}}{c} + \nabla \cdot \vec{\tau}. \qquad (7)$$

Finally, the energy equation must reflect the electrical energy which has invaded the fluid system, both in the form of Ohmic heating and in the energy exchanged between the electromagnetic and flow fields:

$$\rho T \frac{DS}{Dt} = \vec{j} \cdot \vec{E} - \left(q\vec{E} + \vec{j} \times \frac{\vec{B}}{c}\right) \cdot \vec{v} + \nabla \cdot (\vec{v}\vec{\tau}). \qquad (8)$$

Three relationships which are implied in the writing of these equations in this fashion are:[4]

$$\vec{D} = \epsilon \vec{E},$$
$$\vec{B} = \mu \vec{H},$$
$$\frac{1}{\epsilon\mu} = c^2.$$

3. Simplifying Assumptions.

In order to reduce this rather formidable set of equations to one which might be more easily handled, several simplifying assumptions are made. Some justification is listed for each assumption, so that it may be seen that solution validity is not degraded into absurdity.

With regard to the fluid, viscous forces are considered to be many times smaller than the electromagnetic, and they are ignored. This assumption will not produce appreciable error, considering that the magnetic field must be strong enough to produce pressures in excess of the ambient pressure of the free stream. Further, the flow is considered isentropic with no shock effects. Since the flow is in all cases supersonic, again this does not deviate greatly from the actual situation.

[4] Jordan, Electromagnetic Waves and Radiating Systems, Prentice-Hall, New York, 1950

No flow exists in the z-direction, the direction of current flow. Since the main interest in this thesis is in the field-flow interrelationships, rather than in the flow itself, the one-dimensional channel flow approximation is used to describe the two-dimensional flow. This is a standard fluid mechanics technique[5] of sufficient accuracy to justify its use in most nozzle problems. A steady state has been reached in the flow. Finally, the plasma is assumed to behave as a perfect gas, with state and entropy equations of[6]

$$pv = RT \tag{9}$$

$$S = c_v \ln \frac{p}{\rho^{\gamma}} \tag{10}$$

Electrically, the fluid is assumed to have infinite conductivity, since the conductivity of a plasma is extremely high.

4. Consequences of These Assumptions.

The assumption of infinite conductivity within the plasma tremendously simplifies the system of equations 1 through 8 since the currents induced in the conductor through the action of the external field will just balance this field, thus excluding the magnetic vector $\vec{B}$ from the inside of the fluid. This is a restatement of the fact that a perfect conductor is likewise a perfect diamagnet. It is quickly shown[7] from a consideration of equations 1 through 8 at steady state,

[5] Liepman and Puckett, Aerodynamics of a Compressible Fluid, Wiley and Sons, New York, 1947

[6] Kiefer, Kinney and Stuart, Principles of Engineering Thermodynamics, Wiley and Sons, New York, 1954

[7] Fried, Flow through a Plane Magnetic Nozzle, STL Rpt ARL-6-40, 1956

with $\sigma = \infty$ and q = 0, conditions implied by the property of infinite conductivity. Then equation 5 becomes

$$\vec{E} + \vec{v} \times \frac{\vec{B}}{c} = 0$$

or

$$\vec{v} \times \vec{B} = -c\vec{E}.$$

But, within the conductor, $\vec{E}$ does not exist, and therefore

$$\vec{v} \times \vec{B} = 0.$$

This implies that these vectors are parallel or that

$$\vec{B} = \lambda \rho \vec{v}, \tag{11}$$

where λ is a scalar point function defined by this equation.

It is further known from equation 3 that

$$\nabla \cdot \vec{B} = 0, \tag{3}$$

and that at steady state, equation 6 becomes

$$\nabla \cdot (\rho \vec{v}) = 0. \tag{12}$$

Substituting 11 into 3 and expanding

$$\nabla \cdot \lambda \rho \vec{v} = \lambda \nabla \cdot (\rho \vec{v}) + \rho \vec{v} \cdot \nabla \lambda.$$

But, from 12 $$\nabla \cdot (\rho \vec{v}) = 0,$$

and thus $$\nabla \cdot \lambda \rho \vec{v} = \rho \vec{v} \cdot \nabla \lambda.$$

From 3 $$\nabla \cdot \vec{B} = \nabla \cdot \lambda \rho \vec{v} = 0.$$

Therefore, $$\rho \vec{v} \cdot \nabla \lambda = 0,$$

indicating that along any streamline, λ is a constant. But, since $\vec{B}$ approaches zero as x becomes infinite, λ must be zero on all streamlines at infinity. λ is constant along each streamline and is thus zero everywhere in the fluid. From 11, $\vec{B}$ must therefore be zero throughout the fluid.

The use of the steady-state, isentropic, non-viscous, one-dimensional channel flow approximation to describe the fluid flow reduces the

hydrodynamic equations to the simplified forms[8]

$$\text{(mass)} \qquad \rho A \vec{v} = \rho_0 A_0 \vec{v}_0 \tag{14}$$

$$\text{(momentum)} \qquad \rho \vec{v} \frac{dv}{dx} + \nabla p = \vec{j} \times \frac{\vec{B}}{c} \tag{15}$$

$$\text{(energy)} \qquad \frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} = \frac{1}{2} v_0^2 + \frac{\gamma}{\gamma-1} \frac{p_0}{\rho_0} \tag{16}$$

The assumptions used have thus reduced the problem to a consideration of two coupled free-boundary problems with the only common element provided by the surface between them. These problems are essentially the existence of a magnetic field due to a current I, in the presence of a perfect conductor, and the isentropic flow of a non-viscous fluid through a constraining channel.

The common surface is an unknown and will be represented as

$$y = f\,(x)$$

or simply, for brevity

$$y = f. \tag{17}$$

5. The Common Surface and its Imposed Boundary Conditions.

The surface, $y = f$, can be considered in two sections. When $|x| < m$ as shown in Fig. 1, $y = f$ will actually be the locus of all points at which the fluid pressure due to hydrodynamic flow exactly balances the pressure externally applied to the plasma by the magnetic field. When $|x| > m$, $y = y_o$.

This pressure exerted by the magnetic field may be derived from a consideration of the simplified equations at zero velocity[9].

---

[8] Liepman and Roshko, Elements of Gas Dynamics, Wiley and Sons, New York, 1957

[9] Tonks, Particle Transport, Electric Currents, and Pressure Balance in a Magnetically Immobilized Plasma, Phys. Rev. (97), 1443, Jan-Mar 1955

Equation 15 becomes

$$\nabla p = \vec{j} \times \frac{\vec{B}}{c} . \tag{18}$$

Under the steady state assumption, with a direct current, equation 1 is now

$$\nabla \times \vec{B} = \frac{4\pi \vec{j}}{c} . \tag{19}$$

Substituting from 19 for j in 18,

$$\nabla p = \frac{c}{4\pi} (\nabla \times \vec{B}) \times \frac{\vec{B}}{c}$$

or

$$4\pi \nabla p = (\nabla \times \vec{B}) \times \vec{B}. \tag{20}$$

Expanding,

$$4\pi \nabla p = (\vec{B} \cdot \nabla)\vec{B} - \frac{1}{2} \nabla \vec{B}^2 \tag{21}$$

But the gradient of the field is small in the direction of the field and the first term may be neglected. Therefore,

$$\nabla p = -\frac{1}{8\pi} \nabla \vec{B}^2$$

and

$$p = \frac{B^2}{8\pi} \tag{22}$$

This states the pressure exerted by the magnetic field on the fluid in terms of the field intensity.

The relationship of $\vec{B}$ to the current producing the field is given in any basic text on electromagnetics[10], and in CGS units is

$$\vec{B} = \frac{4\vec{I}}{cd} . \quad (23$$

The hydrodynamic pressure at any point in the flow must then be determined. Starting with the energy relationship, equation 16, an equation can be developed which expresses the pressure ratio at any point in a nozzle as an implicit function of the ratio of the nozzle area at that point to the free stream area. If a unit depth in the

---

[10] Jordan, op. cit.

z-direction is considered, this area ratio will reduce to a ratio of nozzle y-coordinates.

If equation 16,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{2}v_0^2 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}, \tag{16}$$

is divided through by the final term, the result is

$$\frac{\gamma-1}{2\gamma} v^2 \frac{\rho_0}{p_0} + \frac{p}{p_0}\frac{\rho_0}{\rho} = \frac{\gamma-1}{2\gamma} v_0^2 \frac{\rho_0}{p_0} + 1. \tag{24}$$

If the sonic velocity relationship[11],

$$a_0^2 = \frac{\gamma p_0}{\rho_0}, \tag{25}$$

is substitued in equation 24, it becomes

$$\frac{\gamma-1}{2}\frac{v^2}{a_0^2} + \frac{p}{p_0}\frac{\rho_0}{\rho} = \frac{\gamma-1}{2}\frac{v_0^2}{a_0^2} + 1. \tag{26}$$

If the first term is multiplied by v_0^2/v_0^2, and the definition of the Mach number,

$$M = \frac{v_0}{a_0} \tag{27}$$

is substituted, the equation becomes

$$\frac{\gamma-1}{2}\frac{v^2}{v_0^2}M^2 + \frac{p}{p_0}\frac{\rho_0}{\rho} = \frac{\gamma-1}{2}M^2 + 1 \tag{28}$$

For a unit depth in the z-direction, the mass equation, 14, may be written as

$$\frac{v}{v_0} = \frac{\rho_0 y_0}{\rho f} \tag{29}$$

with f defined in equation 17. Substituting this, and making use of the isentropic relationship,

$$\frac{\rho_0}{\rho} = \left(\frac{p}{p_0}\right)^{-1/\gamma} \tag{30}$$

to remove the density term, the final result is

$$\frac{\gamma-1}{2}M^2\left(\frac{y_0}{f}\right)^2\left(\frac{p}{p_0}\right)^{-2/\gamma} + \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\gamma-1}{2}M^2 + 1 \tag{31}$$

---

[11] Liepman and Roshko, op. cit.

It is noted that, if incompressible flow should be under consideration, the derivation is much simpler. In this case, the energy equation reduces to Bernoulli's equation,

$$p + \frac{1}{2}\rho v^2 = p_o + \frac{1}{2}\rho v_o^2 . \qquad (32)$$

Dividing through by p_o, and again using equation 14 to introduce the nozzle shape and eliminate the velocity,

$$\begin{aligned} \frac{p}{p_o} &= 1 + \frac{\rho}{2p_o}\left(v_o^2 - v^2\right) \\ &= 1 + \frac{\rho v_o^2}{2p_o}\left(1 - \frac{v^2}{v_o^2}\right) \\ &= 1 + \frac{\rho v_o^2}{2p_o}\left[1 - \left(\frac{y_o}{f}\right)^2\right]. \end{aligned} \qquad (33)$$

This is not only a simpler derivation, but because of the explicit relationship between pressure and nozzle shape, would probably simplify some of the later discussion.

The boundary conditions imposed by this problem must now be considered. Since the fluid is confined everywhere by the surface of the nozzle,

$$\vec{v} \cdot \vec{n} = 0 \qquad (34)$$

everywhere on $y = f$. Further, since there is no magnetic field within the fluid,

$$\vec{B} \cdot \vec{n} = 0 \qquad (35)$$

along the same surface. Finally, since the magnetic field is shaping the nozzle in the region where $-m \leq x \leq m$ (See Fig. 1), the magnetic pressure must equal the fluid pressure at every point in this region in order to establish an equilibrium surface. Thus,

$$\frac{B^2}{8\pi} = p \qquad (36)$$

at every point on the surface where $-m \leq x \leq m$.

In the discussion of the boundary conditions and in the further mathematical development, it is convenient to describe the magnetic field by means of a vector magnetic potential $\vec{A}$ [12], such that

$$\vec{B} = \nabla \times \vec{A} . \tag{37}$$

The symmetry of the problem indicates that $\vec{A}$ will have only one non-zero component, $\vec{A}_z$, permitting it to be treated as a scalar quantity. This potential must satisfy Laplace's equation outside the source wire and Poisson's equation within it. Thus, outside,

$$\nabla^2 A = 0 \tag{38}$$

and inside,

$$\nabla^2 A = -\frac{4\pi}{c} I \tag{39}$$

Introducing a Dirac delta function, defined by

$$\delta(x) = 0, \quad x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

these two expressions may be written as a single term

$$\nabla^2 A = -\frac{4\pi}{c} I \; \delta(x) \; \delta[y-(y_0+r)] \tag{40}$$

Relating the boundary conditions expressed in equations 35 and 36 to this magnetic potential, it is seen that they are satisfied if

$$A = 0$$

on y = f or, expressed in another notation,

$$A[x, f] = 0; \tag{41}$$

and, if

$$|\nabla A|^2 = 8\pi p \tag{42}$$

on $y = f$, $-m \leqslant x \leqslant m$.

---

[12] Jordan, op. cit.

The boundary conditions are complicated by the fact that this is a free boundary problem; that is, the boundary is unknown, and in fact, it is the objective of this thesis to determine this boundary for a given set of conditions.

There are, however, some general restrictions which may be placed on the boundary which will aid in visualizing a solution. The symmetry of the nozzle about the x-axis has already been mentioned. It is further apparent from Fig. 1 that, since the process is assumed isentropic, the nozzle is likewise symmetrical about the y-axis. The nozzle cannot have a sharp corner at $x = \pm m$, since this would imply the existence of an infinite magnetic pressure at that point, a point patently inconsistent with the equality of magnetic and hydrodynamic pressures stated in equation 36. Thus the slope of the nozzle surface must be zero at $x = \pm m$. This slope must likewise, from symmetry considerations, be zero at $x = 0$. It is then apparent from the mean value theorem of the calculus that inflection points exist between $x = 0$ and $x = \pm m$, and that the slope of the nozzle is a maximum at these points.

6. Current Criticality and its Effect on Nozzle Shape.

From a further consideration of the implications of the equality of magnetic and hydrodynamic pressures at all points along the nozzle, it can be seen that some minimum critical value of current must be used or the magnetic pressure even at the closest point of flow to current will not be sufficient to permit this equality to be established. For any current I through the wire, the maximum field intensity along the channel wall will occur at the point $(0, y_o)$, and can be expressed from equation 23 as

$$B = \frac{4I}{cr} . \qquad (43)$$

If I is taken as zero, the plasma flow would proceed as though it were ordinary hydrodynamic flow through a constant area channel. As I increases, B will likewise increase, but the flow will remain unaffected until the magnetic pressure becomes equal to p_o. Thus I must have at least a value so that

$$\frac{16 I_c^2}{8\pi c^2 r^2} = p_o$$

or

$$I_c = cr\sqrt{\frac{\pi p_o}{2}}\,. \tag{44}$$

This is referred to as the critical current. For values of I greater than I_c , the flow will be nozzled down as much as is required to bring the magnetic and hydrodynamic pressures into equilibrium. When equilibrium is reached the nozzle shape is established.

It is further apparent that, for any current value in excess of the critical, the magnetic and free stream hydrodynamic pressure will be equal at the breakaway points, $x = \pm m$. Representing the distance from the wire to the breakaway point by d_m, this may be stated as

$$\frac{16 I^2}{8\pi c^2 d_m^2} = p_o,$$

from which

$$d_m^2 = \frac{2I^2}{\pi c^2 p_o}\,.$$

From the Pythagorean theorem, it is also true that

$$d_m^2 = r^2 + m^2.$$

Thus,

$$m^2 = \frac{2I^2}{\pi c^2 p_o} - r^2$$

But any current in excess of the critical value may be represented as some multiple, n, times the critical where $n > 1$. Thus,

$$m^2 = \frac{2n^2 I_c^2}{\pi c^2 p_o} - r^2.$$

Substituting from equation 44,

$$m^2 = \frac{2n^2c^2r^2\pi p_0}{2\pi c^2 p_0} - r^2$$

$$= (n^2-1)\,r^2 . \tag{45}$$

Hence, the nozzle length is directly connected to the relationship between current used and the critical current for the system.

7. Discussion of a Mathematical Solution.

The task of solving these equations is a formidable, if not impossible, one for the general case. Hence, an analytic solution in closed form has not been obtained, but an approach which may be useful in finding the solution in certain specific cases is given.

This involves the variational principle of the calculus of variations. Dr. Fried uses it[13] in a similar case where he has employed a non-dimensional magnetic potential.

Let the right hand side of equation 40 be represented by a single symbol, say g.

$$g = -\frac{4\pi}{c} I \;\delta(x)\; \delta[y-(y_0+r)]. \tag{46}$$

Then

$$\nabla^2 A = g. \tag{47}$$

An energy integral for the system may be defined for any pair of functions, A(x, y) and f(x), as

$$E(A,f) = \int_R dx\,dy \left(\frac{c^2}{32I^2}|\nabla A|^2 + \frac{gc}{4I} A\right) + \int_{-\infty}^{\infty} dx\; h[f(x)] \tag{48}$$

where R is the region of the xy-plane above the curve y = f, and where h $[f(x)]$ is a function of the pressure at any point on the nozzle satisfying the relationship

$$\frac{dh}{df} = -\frac{\pi c^2}{4I^2} p\,, \tag{49}$$

p being developed from equation 31. This term will be defined as P/2.

---

[13] Fried, op. cit.

The first term in equation 48 is an integral of $\frac{B^2}{8\pi}$ over the region where it is not zero, while the latter term integrates p throughout the volume occupied by the fluid, thus comprising an energy integral for the system.

From equation 41, it is known that A and f, although both are arbitrary, must satisfy

$$A[x, f] = 0. \tag{41}$$

Then small variations, δA and δf, give a variational relationship,

$$\delta E = \int_R dx\, dy \left(\frac{c^2}{32I^2} \nabla A \cdot \nabla \delta A + \frac{cg}{4I} \delta A\right) - \int_{-\infty}^{\infty} dx\, \delta f \left(\frac{c^2 |\nabla A|^2}{32 I^2} + \frac{cg}{4I} A\right) - \int_{-\infty}^{\infty} dx\, \delta f \frac{p}{2}\, . \tag{50}$$

These terms represent respectively the variation of the first term of the energy integral with variation in A; the variation of the same term with variation in f; and the variation of the last term of the energy integral with f only, since it is independent of A. The negative sign of the last two terms derives from a decrease in the value of the integral concerned when a positive (increasing) variation is permitted in f. The second integral in equation 50, although coming from the area term of the original energy integral, is now integrated with respect to x only, since y, which is equal to f, has now been replaced in the increment by δf.

If δf is considered to mean small variations in the surface bounding the fluid flow, it will vanish when f coincides with y_o in the fixed portion of the system. As a result, the last two integrals need be evaluated only from -m to m, since they will be zero outside this range.

The first integral in equation 50 may be further treated by applying the vector identity[14,15]

$$\nabla A \cdot \nabla \delta A = \nabla \cdot (\delta A \nabla A) - \delta A \nabla^2 A \tag{51}$$

From this,

$$\int_R dx\, dy \left(\frac{c^2}{32I^2} \nabla A \cdot \nabla \delta A + \frac{cg}{4I}\, \delta A\right) = \int_R dx\, dy \left(\frac{c^2}{32I^2}\left[\nabla \cdot (\delta A \nabla A) - \delta A \nabla^2 A\right] + \frac{cg}{4I}\, \delta A\right). \tag{52}$$

Combining terms, expanding, treating δA as a constant value, and applying the divergence theorem to the result, it is further reduced to

$$\int_R dx\, dy \left[\delta A \left(\frac{cg}{4I} - \frac{c^2}{32I^2} \nabla^2 A\right)\right] - \frac{c^2}{32I^2} \int_{-\infty}^{\infty} ds\, (\vec{n} \cdot \nabla A)\, \delta A \tag{53}$$

where ds is the element of arc length along y = f, and $\vec{n}$ is the unit normal to s inward on R.

Again recalling that, from equation 41,

$$A[x, f] = 0, \tag{41}$$

it is seen that along y = f,

$$\delta A + \frac{\partial A}{\partial f}\, \delta f = 0,$$

or that

$$-\delta A = \frac{\partial A}{\partial f}\, \delta f. \tag{54}$$

Considering the second integral of equation 53, it is noted that the term in parenthesis may be rewritten as

$$\vec{n} \cdot \nabla A \equiv \frac{\partial A}{\partial n} \tag{55}$$

Substituting these two into the second integral, and recalling the remarks on the limits of integration above, it may now be written

$$\int_{-\infty}^{\infty} ds\, (\vec{n} \cdot \nabla A)\, \delta A = \int_{-m}^{m} \frac{ds}{dx}\, dx\, \frac{\partial A}{\partial n}\, \frac{\partial A}{\partial f}\, \delta f, \tag{56}$$

---

[14] Wiley, Advanced Engineering Mathematics, McGraw-Hill, New York, 1951

[15] Brand, Vector and Tensor Analysis, Wiley and Sons, New York, 1947

With the expressions developed in equations 53 and 56 substituted into equation 50, the variational form of the energy integral, it can now be rewritten as

$$\delta E = \int_R dx\, dy \left[\delta A\left(\frac{cg}{4I} - \frac{c^2}{32I^2}\nabla^2 A\right)\right] + \int_{-m}^{m} dx\, \delta f\left[\frac{\partial A}{\partial n}\frac{c^2}{32I^2}\frac{\partial A}{\partial f}\frac{ds}{dx} - \frac{c^2}{32I^2}|\nabla A|^2 - \frac{cg}{4I}A - \frac{P}{2}\right] \tag{57}$$

Since $A[x, f] = 0$, the gradient of A must be in the direction of the normal to y = f, and therefore

$$\vec{n}\cdot\nabla A \equiv \frac{\partial A}{\partial n} = |\nabla A|. \tag{58}$$

Considering

$$\begin{aligned}\frac{\partial A}{\partial f}\frac{ds}{dx} &= \frac{\partial A}{\partial y}\frac{ds}{dx}\\ &= \frac{\partial A}{\partial y}\frac{\sqrt{(dx)^2+(dy)^2}}{dx}\\ &= \sqrt{\left(\frac{\partial A}{\partial y}\right)^2\left[\frac{(dx)^2+(dy)^2}{(dx)^2}\right]}\\ &= \sqrt{\left(\frac{\partial A}{\partial y}\right)^2+\left(\frac{\partial A}{\partial x}\right)^2}\\ &= |\nabla A|,\end{aligned} \tag{59}$$

it is seen that

$$\frac{\partial A}{\partial n}\frac{\partial A}{\partial f}\frac{ds}{dx} = |\nabla A|^2. \tag{60}$$

Now, since

$$g = -\frac{4\pi}{c} I\, \delta(x)\,\delta(y - [y_0 + r]) \tag{46}$$

vanishes on y = f, the end result is

$$\delta E = \int_R dx\, dy\, \delta A\left(\frac{cg}{4I} - \frac{c^2}{32I^2}\nabla^2 A\right) + \int_{-m}^{m} dx\, \frac{\delta f}{2}\left\{\frac{c^2}{32I^2}|\nabla A|^2 - P\right\}. \tag{61}$$

This will vanish, satisfying the variational principle, if and only if A and f satisfy the equations 41 and 42.

These results may be further applied to this problem by noting from equation 41 that A is a functional of f; that is, A depends in a unique

manner on f. Then, if f were chosen on any convenient basis, A would be determined by equations 41 and 61.

Consequently, the energy integral becomes a functional of f alone, rather than of f and A, and setting

$$\delta E = 0, \tag{62}$$

as prescribed in the variational principle, should develop an equation which is in terms of f, and which would be satisfied by the f which together with its associated A, satisfies equations 41 and 42.

At this point, consider the introduction of a Green's function of two arbitrary position vectors of the region R,

$$G(\vec{r}_1, \vec{r}_2) \equiv \frac{1}{2\pi} \ln(\vec{r}_1 - \vec{r}_2). \tag{63}$$

Because

$$\nabla^2 G(\vec{r}_1, \vec{r}_2) = \delta(\vec{r}_1 - \vec{r}_2), \tag{64}$$

it is possible to use a function

$$\phi(\vec{r}) = -\frac{4\pi}{c} I G(\vec{r}_1, \vec{r}_0) \tag{65}$$

where

$$\vec{r}_0 = \vec{j}\,(y_0 + r) \tag{66}$$

as a solution for

$$\nabla^2 A(\vec{r}) = g(\vec{r}) = -\frac{4\pi}{c} I \delta(\vec{r}_1 - \vec{r}_0), \tag{67}$$

which restates in vector form equation 40, since the Dirac delta function will be non-zero only at the position of the current carrying conductor.

Reconsider now the statement of equation 48. Since the second term of the integral is already a function of f alone, only the first term will be discussed at this point.

This term may now be written, considering the area summation as the summing of all position vectors, $\vec{r}_1$, as

$$\int_R d\vec{r}_1 \left(\frac{c^2}{32 I^2} |\nabla A|^2 + \frac{cg}{4I} A \right). \tag{68}$$

However, employing the vector identity

$$\nabla\cdot(A\nabla A = |\nabla A|^2 + A\nabla^2 A, \tag{69}$$

this may be expanded to

$$\int_R d\vec{r}_1 \left(\frac{c^2}{32I^2}[\nabla\cdot(A\nabla A) - A\nabla^2 A] + \frac{c}{4I}\, gA\right). \tag{70}$$

But, from equation 46,

$$\nabla^2 A = g \,. \tag{46}$$

Collecting terms, and noting that, from the symmetry of the problem, the integral of the first term vanishes, the integral becomes

$$\int_R d\vec{r}_1 \left(-\frac{cg}{32I^2} A + \frac{cg}{4I} A\right) = \frac{c}{4I}\left(1 - \frac{c}{8I}\right)\int_R d\vec{r}_1\, gA \tag{71}$$

But because of the definition of ϕ(r) above,

$$g = \nabla^2\phi, \tag{72}$$

and setting

$$\frac{c}{4I}\left(1 - \frac{c}{8I}\right) = k, \tag{73}$$

the integral now becomes

$$k\int_R d\vec{r}_1\, A\nabla^2\phi. \tag{74}$$

If the expression

$$\phi\nabla^2 A + \nabla A\cdot\nabla\phi$$

be added and subtracted from the integrand, the integral develops into

$$k\int_R d\vec{r}_1 \left(\phi\nabla^2 A + \nabla A\cdot\nabla\phi + A\nabla^2\phi - \nabla A\cdot\nabla\phi - \phi\nabla^2 A\right). \tag{75}$$

This may be manipulated into

$$\begin{aligned} &k\int_R d\vec{r}_1 [\phi\nabla^2 A + \nabla\cdot(A\nabla\phi - \nabla\cdot(\phi\nabla A)] \\ &= k\int_R d\vec{r}_1 [\phi\nabla^2 A + \nabla\cdot(A\nabla\phi - \phi\nabla A)] \end{aligned} \tag{76}$$

The integration of the first term gives

$$-k\pi\,\phi(\vec{r}_0). \tag{77}$$

The second term vanishes over the area integration, and the third term may be transformed by means of the divergence theorem to

$$k\int_{-\infty}^{\infty} ds\, \phi\,\frac{\partial A}{\partial n}, \tag{78}$$

by use of equation 58 and recalling that $\vec{n}$ has been defined as a unit normal inward on R.

If ds is expressed in terms of dx, the first term of equation 48 becomes

$$E = -k\pi\,\phi(\vec{r}_0) + \int_{-\infty}^{\infty} dx \left\{ k\left[1 - \left(\frac{df}{dx}\right)^2\right]^{1/2} \phi\, \frac{\partial A}{\partial n} \right\}. \tag{79}$$

Combining this with the second term of equation 48, the energy integral may be written

$$E = -k\pi\,\phi(\vec{r}_0) + \int_{-\infty}^{\infty} dx \left\{ k\left[1 - \left(\frac{df}{dx}\right)^2\right]^{1/2} \phi\, \frac{\partial A}{\partial n} + h \right\} \tag{80}$$

Although the first term is infinite in the limit, it is independent of f and thus, when the variational principle is applied, may be ignored as it will contribute nothing to δE. Thus, formally, the problem has been reduced to a dependence on f alone, and when δE is set equal to zero, an exact equation for f will emerge. Practically, however, the dependence of A on f, as defined by equation 61, is so complex in general that the expression for δE cannot be written.

8. Numerical Solutions for Specific Cases.

Because of the complexity of the relationships necessary for analytic solution, the problem was attacked numerically, using the Model NCR-102A computer facility available at the U. S. Naval Postgraduate School.

In this attack, equation 31 was first solved. There are four variables in this equation. The specific heat ratio and Mach number were arbitrarily fixed at one of five preselected values. Mach numbers began at 5.0 and proceeded in steps of five to a value of 25.0. Values of the specific heat ratio ran from 1.30 to 1.50 in increments of 0.05. All possible combinations of these two parameters were eventually used

in the solution. This reduced the equation to one in which only two variables exist for any specific case. Because of its exponents, the pressure ratio was selected as the independent variable, with values from 1.000 to 5.000. This permitted solution for the y-coordinate ratio corresponding to a given pressure ratio. An interpolation routine in the computer then converted the ratio of y-coordinates to the independent variable, tabling the pressure ratios corresponding to preselected values of this parameter. This approach was used because the use of pressure ratio as the independent variable permitted a simpler solution of equation 31. The final results were given for preselected values of the y-coordinate of the nozzle, allowing simpler graphing, and, to the author, permitting a more readily understood and quickly prepared tabulation of nozzle shape data. The values of pressure ratio corresponding to the nozzle y-coordinate for the combinations of M and γ are given in Table I.

With this accomplished, for a combination of Mach number and specific heat ratio chosen from those used, the fluid pressure at any point along the nozzle can be determined in terms of the free stream pressure if the y-coordinate of the nozzle at that point as compared to y_o is known. Since, at every point along the nozzle, the magnetic pressure must equal the fluid pressure, the magnetic pressure is known in its relationship to the free stream pressure once a y-coordinate is picked.

The magnetic pressure is related to the current in the conductor through equations 22 and 23,

$$p_m = \frac{B^2}{8\pi}, \tag{22}$$

$$B = \frac{4I}{cd}. \tag{23}$$

in the solution. This reduced the equation to one in which only two variables exist for any specific case. Because of its exponents, the pressure ratio was selected as the independent variable, with values from 1.000 to 5.000. This permitted solution for the y-coordinate ratio corresponding to a given pressure ratio. An interpolation routine in the computer then converted the ratio of y-coordinates to the independent variable, tabling the pressure ratios corresponding to preselected values of this parameter. This approach was used because the use of pressure ratio as the independent variable permitted a simpler solution of equation 31. The final results were given for preselected values of the y-coordinate of the nozzle, allowing simpler graphing, and, to the author, permitting a more readily understood and quickly prepared tabulation of nozzle shape data. The values of pressure ratio corresponding to the nozzle y-coordinate for the combinations of M and γ are given in Table I.

With this accomplished, for a combination of Mach number and specific heat ratio chosen from those used, the fluid pressure at any point along the nozzle can be determined in terms of the free stream pressure if the y-coordinate of the nozzle at that point as compared to y_o is known. Since, at every point along the nozzle, the magnetic pressure must equal the fluid pressure, the magnetic pressure is known in its relationship to the free stream pressure once a y-coordinate is picked.

The magnetic pressure is related to the current in the conductor through equations 22 and 23,

$$p_m = \frac{B^2}{8\pi}, \tag{22}$$

$$B = \frac{4I}{cd}. \tag{23}$$

Therefore, if a known current flows in the conductor, the distance from the conductor at which the magnetic pressure is equal to the stream pressure for a certain nozzle y-coordinate may be calculated. If the conductor is located a specified distance from the outer wall of the channel, then all conditions are known to completely determine the nozzle shape.

Thus, to develop the resulting nozzle shapes for specific conditions, current flowing in the conductor must be known. As described above, a certain minimum current must flow, else the fluid is undisturbed. This critical current was shown in equation 44 to be

$$I_c' = cr\sqrt{\frac{\pi p_0}{2}}. \tag{44}$$

If the current is chosen in multiples of this critical current, as was done for this thesis, three advantages accrue. First, all troublesome constants disappear. Secondly, the equations reduce to simple expressions which are ideally suited for digital computer work. Finally, the work is done non-dimensionally and thus wider applicability of the results are gained. The currents used here are respectively, $2I_c$, $3I_c$, $4I_c$, and $5I_c$.

The final item which affects a specific nozzle shape is the distance r, the separation between the current carrying conductor and the channel wall. This influences the magnetic pressure, since the absolute value of the critical current depends on this distance. It also has a more direct influence on the nozzle shape since the abscissa of any point on the nozzle will be the third side of a right triangle whose hypoteneuse, d, and side, r, are known. The advantages of non-dimensionality are again applicable and, therefore, in this problem, the r's used are $y_o/5$, $y_o/4$, and $y_o/3$.

A variation of p_o was not considered in this thesis. Since the currents were used in multiples of the critical current, which is a function of p_o, the variation of current parallels a variation of pressure. Thus, if current were held constant and pressure changed by a factor, the number of critical currents represented would be changed by the inverse square root of that factor, and the new nozzle would be shaped exactly as a nozzle whose forming current was represented by this number of critical currents with the initial pressure.

As a corollary to this, the shaping factor of the nozzle may be expressed as the ratio,

$$\omega = \frac{I}{\sqrt{p_o}} . \tag{81}$$

If this factor is held constant, the nozzle shape for a given Mach number, specific heat ratio, and conductor-wall separation will not change.

By studying the effects of varying each parameter in turn for constant combinations of the others, both general qualitative statements of effect on nozzle shape and the actual shapes for the given conditions can be realized.

In arranging the work for digital computation, some multiple of the critical current was assumed to flow.

$$I = n I_c = ncr \sqrt{\frac{\pi p_o}{2}} . \tag{82}$$

Therefore,

$$\begin{aligned} B &= \frac{4I}{cd} \\ &= \frac{4nr\sqrt{\frac{\pi p_o}{2}}}{d} , \end{aligned} \tag{23}$$

and

$$p_m = \frac{B^2}{8\pi} \tag{22}$$

$$= \frac{16 n^2 r^2 \pi p_o}{16 \pi d^2},$$

or

$$\frac{p_m}{p_o} = \frac{n^2}{(d/r)^2}. \tag{83}$$

But from the boundary conditions, at any point along the nozzle,

$$p_m = p_f \tag{84}$$

or

$$\frac{p_m}{p_o} = \frac{p_f}{p_o}.$$

Therefore,

$$\frac{n^2}{(d/r)^2} = \frac{p_f}{p_o} \tag{85}$$

$$\frac{d}{r} = \frac{n}{\sqrt{p_f/p_o}}. \tag{86}$$

This equation is the essence of the point-by-point calculation of a nozzle shape for a particular Mach number, specific heat ratio, current, and distance of conductor from the channel wall.

For given values of M and γ, let a value of $\left(\frac{f}{y_o}\right)$ be selected. This accomplishes two things. The value, $\left(\frac{p_f}{p_o}\right)$, is fixed, and a straight line locus of points in the fluid at which that pressure ratio may exist is established. With a fixed pressure ratio and current, the distance at which a magnetic pressure equal to the fluid pressure may exist can be calculated in terms of r. Since r is known, a circle of now known radius is the locus of points at which the equal magnetic pressure exists. The intersections of these two loci establish the points on the nozzle.

To calculate the nozzles for specific conditions, the values of $\left(\frac{p_f}{p_o}\right)$ previously tabulated for those conditions were fed to the computer, along with the number of critical currents in each of the currents used. The indicated division was performed and the $\left(\frac{d}{r}\right)$ ratio was obtained. These ratios are tabulated in Tables II through VI for each

of the conditions outlined. With these values, graphs of the nozzle shapes were made by constructing the loci as described above. Graphs were not made for each of the sets tabulated. This was unnecessary since those graphs shown indicate the effect of a change in one parameter with the others constant. This effect is similar, as shown by the values tabulated, no matter what values are assigned to the fixed parameters.

Each nozzle has a minimum diameter. This occurs when the circular locus of equal magnetic pressure is just tangent to the straight line. That such a point exists is evident from the following consideration. As the y-coordinate of the nozzle decreases, the fluid pressure increases. Consequently, the magnetic pressure must likewise increase to maintain equilibrium. This demands a smaller diameter for the circular locus. Thus a point will be reached where the diameter of the circular locus is just equal to the sum of the conductor-wall separation and the fluid boundary displacement from the wall, and here the nozzle minimum will occur. The (d/r) value of these minima are included in Tables II through VI, for values of $r = 0.2y_o$ for all conditions, and $r = 0.25y_o$ and $0.33y_o$ for all $\gamma = 1.30$.

In actually determining the point at which the minimum occurred, it was helpful to construct a table giving the vertical distance from the conductor to each of the y-coordinates used in the problem. If the value of $\left(\frac{d}{r}\right)$ as calculated above exceeded this distance for a specific y-coordinate, then the two loci intersected. If the reverse were true, there was no intersection, and hence the nozzle surface did not exist at that y-coordinate. A comparison of this table with the tables of $\left(\frac{d}{r}\right)$ constructed for the various conditions gave the crossover values, and

a cross interpolation provided the actual minimum point. This table is Table VII.

It may also be noted from equation 86 that at the breakaway point in each case, the ratio, $\left(\frac{d}{r}\right)$, is exactly equal to the number of critical currents, n, since at that point, the ratio, $\left(\frac{p_f}{p_o}\right)$, is unity.

Consider now the qualitative effects of variation of these parameters singly with all others held constant.

As seen from Fig. 2, or by noting the values in Tables II through VI, an increase in current invariably deepens and lengthens the nozzle. This is predictable, since the magnetic pressure is directly proportional to the square of the current, and thus, at any point, the fluid must be nozzled down to increase the hydrodynamic pressure for equilibrium.

As seen from equation 86, the distance from the conductor to the breakaway point is always

$$d_m = nr$$

where n is the number of critical currents flowing in the conductor, $n > 1$. Equation 45 also points out that the abscissa of either breakaway point is, in magnitude

$$(n^2-1)^{1/2}\, r.$$

If the length of the nozzle is defined as the distance between breakaway points, the length in this case is

$$l_0 = 2r\sqrt{n^2-1}. \qquad (87)$$

If the current is changed by a factor α, the number of critical currents is likewise changed by α, and the new nozzle length would therefore be

$$l_1 = 2r\sqrt{\alpha^2 n^2-1}. \qquad (88)$$

Thus for a change of current by a factor α, the nozzle length is changed by a factor,

$$\frac{l_1}{l_0} = \sqrt{\frac{\alpha^2 n^2 - 1}{n^2 - 1}}, \qquad (89)$$

indicating the change in nozzle length depends not only on the change in current, but also on the amount of current initially flowing.

Defining nozzle depth as the distance, yo - f, at the throat, the change of depth lends itself to no such simple prediction formula because of the complex relationship between the nozzle y-coordinate ratio and the pressure ratio expressed in equation 31, and is best left to a numerical calculation in each case.

If Mach number of the flowing fluid is the variable, some interesting results occur. First, the breakaway point is located at the same spot, regardless of Mach number, since this parameter can influence the radius of the circular locus only through the pressure ratio, and this is unity at the breakaway point, regardless of Mach number. As Mach number increases, the nozzle depth likewise increases, because with an increase in Mach number, the ratio of y-coordinates necessary to achieve a given pressure ratio decreases, as may be seen by a comparison of the data in Table I for various Mach numbers. Again, the complexity of equation 31 precludes setting up a simple expression for predicting depth changes. The most interesting aspect of variation of Mach number, however, lies in the fact that changes in Mach number for the lower values used cause only small disturbances in the nozzle shape and changes occurring above the Mach of 15 cause almost no variation, as shown in Table I. Fig. 3 shows the result of this variation.

When γ, the specific heat ratio of the flowing fluid, is changed, the breakaway point again remains unaffected for the same reasons discussed in the case of Mach number variation. As seen in Table I by comparing the pressure ratios for a given y-coordinate ratio at various values of specific heat ratio, an increase in γ necessitates a larger y-coordinate to realize a given pressure ratio. As a result, the depth of the nozzle decreases with an increase in γ. This change in depth is small but noticeable. Again it is unpredictable. The effect of changes in γ are shown in Fig. 4.

If the distance r between the conductor and the channel wall is changed, with the absolute magnitude of current constant, a double effect is introduced. Since the value of the critical current is dependent on r, the number of critical currents flowing varies inversely with the change in r. And since r has been changed, the value of d, must change, since the $\left(\frac{d}{r}\right)$ ratio will still have the same value. Thus, if r is varied by a factor β,

$$r_1 = \beta r_0$$

$$I = \frac{n}{\beta} c r_1 \sqrt{\frac{\pi p_0}{2}} . \qquad (90)$$

Applying equation 86,

$$\frac{d}{r_1} = \frac{n/\beta}{\sqrt{p_f/p_0}} \qquad (91)$$

and at the breakaway point,

$$d = \frac{n}{\beta} r_1 = n r_0 . \qquad (92)$$

Thus, an increase in r causes a decrease in nozzle length since the distance from the conductor to the breakaway point remains constant, but the side r of the right triangle has increased, necessitating a decrease in m.

The length of the nozzle is now

$$l_1 = 2\sqrt{n^2 r_0^2 - r_1^2} = 2\sqrt{n^2 r_0^2 - \beta^2 r_0^2}$$
$$= 2 r_0 \sqrt{n^2 - \beta^2} . \tag{93}$$

Since

$$l_0 = 2 r_0 \sqrt{n^2 - 1} , \tag{94}$$

it is quickly seen that a change in r by a factor β causes a change in nozzle length by a factor of

$$\frac{l_1}{l_0} = \sqrt{\frac{n^2 - \beta^2}{n^2 - 1}} . \tag{93}$$

This is in accord with the conclusion mentioned earlier, since if β is greater than unity the numerator of this expression is less than the denominator. This demonstrates the inverse relationship between conductor-wall separation and nozzle length. It is also to be noted that the value of β may not exceed that of n. When these two are just equal, the current flowing in the conductor will be exactly one critical current. Any increase in β beyond this value will result in an undisturbed flow.

The nozzle depth is likewise diminished when r is increased. Consider that the y-coordinate of the nozzle throat remains constant when r is increased. The magnetic pressure at that point will drop, since the distance from the conductor to that point has increased. But, for fixed Mach number and specific heat ratio, the hydrodynamic pressure is a function of the y-coordinate of the nozzle only. Therefore, to establish equilibrium, the throat must expand to reduce the hydrodynamic pressure, and the depth of the nozzle is diminished. These results are illustrated in Fig. 5.

If, in lieu of holding the absolute value of the current constant, the number of critical currents is made the constant factor, just the reverse occurs. If r changes by a factor, θ , so that

$$r_1 = \theta r_0, \tag{94}$$

the distance between the conductor and the breakaway point will still be, from equation 86,

$$\frac{d}{r_1} = n, \tag{86}$$

since n is a constant. The nozzle length is then

$$l_1 = 2r_1\sqrt{n^2-1}\,. \tag{95}$$

Since the initial length was

$$l_0 = 2r_0\sqrt{n^2-1}, \tag{96}$$

it is quickly seen that

$$\frac{l_1}{l_0} = \frac{r_1}{r_0} = \theta, \tag{97}$$

Thus the nozzle length is increased by the same factor as the distance r.

Nozzle depth likewise increases. This can be seen from a consideration of the fact that the absolute current has increased by the same factor as the distance r, while the distance d, which is the sum of $(y_0 - f)$ and r, is increased by a smaller factor, since y_0 is a constant. Thus, since the magnetic pressure is proportional directly to the square of the current, and inversely to the square of the distance d, it is seen that the magnetic pressure at the position of the original throat has increased. The nozzle throat must then contract to increase the hydrodynamic pressure and establish equilibrium again. Fig. 6 is a graphical presentation of this effect.

9. Conclusions.

As a result of the solutions assembled and discussed in Section 8,

it is concluded that, in a magnetic nozzle of the type described herein, the most effective method of changing nozzle shape is by varying the current. Variation in Mach number of the flowing plasma or its specific heat ratio has too little effect to be of much use. Nozzle length is completely unaffected. Beyond a Mach number of 15, the nozzle shape becomes nearly independent of the value of this parameter. These would also be considerably more difficult to control than the current. The variation of distance would probably be limited by the physical size of the system. While it is actually the ratio $\left(\frac{I}{\sqrt{p_o}}\right)$ which is the nozzle shaping factor, p_o would be much more difficult to vary effectively than I. For these reasons, current variation is the most effective single factor in shaping the nozzle.

It is further concluded that for extensive work in this field, much more comprehensive tables will be needed. While the values taken herein provide indications of trends in the system, they would hardly suffice for much work where the parameters differed from those used in this thesis. Because of all the non-linearities involved, linear interpolation between tables would probably not be satisfactory, and extrapolation would be meaningless. For each new combination of parameters involved, the complete solution would have to be made, as no solution in closed form was developed.

Mention is made of two mathematical methods which might prove useful in further attacks on this type of problem. While both of these were considered by the author in his researches, lack of time prevented extensive development of them. The first is the method of characteristics

which is expanded by Loughhead[16], based on the work done by Courant, Issacson and Rees[17]. The numerical work in this scheme is very involved, and a discussion of it with the director of the computer facility used convinced the author that the computations would be impossible to do in the time available.

The second is the method of the hypercircle, developed by Synge[18], which has proved a powerful tool in the solution of types of differential equations which were insoluble by other means. Again, the complexity of the calculations deterred the author from pursuing this path.

Finally, it must be mentioned that in the same case with plasma flow velocity zero, an analytic solution for the shape of the nozzle has been made by Fried[19], using the technique of conformal mapping as suggested by Lamb[20]. In this solution, the free boundary can be mapped into a fixed boundary, since, when the velocity is zero, the pressure throughout the plasma is constant. This technique does not improve the situation when the velocity is not zero, since the boundary after mapping is also still unknown. Another interesting aspect of the zero velocity case is the study of the magnetic piston. Under this velocity

---

[16] Loughhead, Solution of Problems Involving the Hydromagnetic Flow of Compressible Ionized Fluids, Phys. Rev. (99) 1678, Jul-Sep 1955

[17] Courant, Issacson, and Rees, On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences, Comm. P. Appl. Math. (5) 243, 1952

[18] Synge, The Hypercircle in Mathematical Physics, Cambridge Press, Cambridge, 1957

[19] Fried, op. cit.

[20] Lamb, Hydrodynamics, University Press, Cambridge, 1930

condition, the throat diameter can be made zero by a magnetic field of sufficient intensity.

Further studies in this field that suggest themselves include the extension to a flow which is bounded in the z-direction and which is nozzled by the current flowing in a circular coil about the fluid. This could be further extended to current flowing in a solenoid. The problem of nozzle shape when fluid conductivity is not infinite is a more practical, but also a considerably more difficult one.

Variation of Nozzle Shape with Change in Current

Constant values:

Mach No. = 5

Specific Heat Ratio = 1.3

Conductor - Wall separation = 0.2 y_0

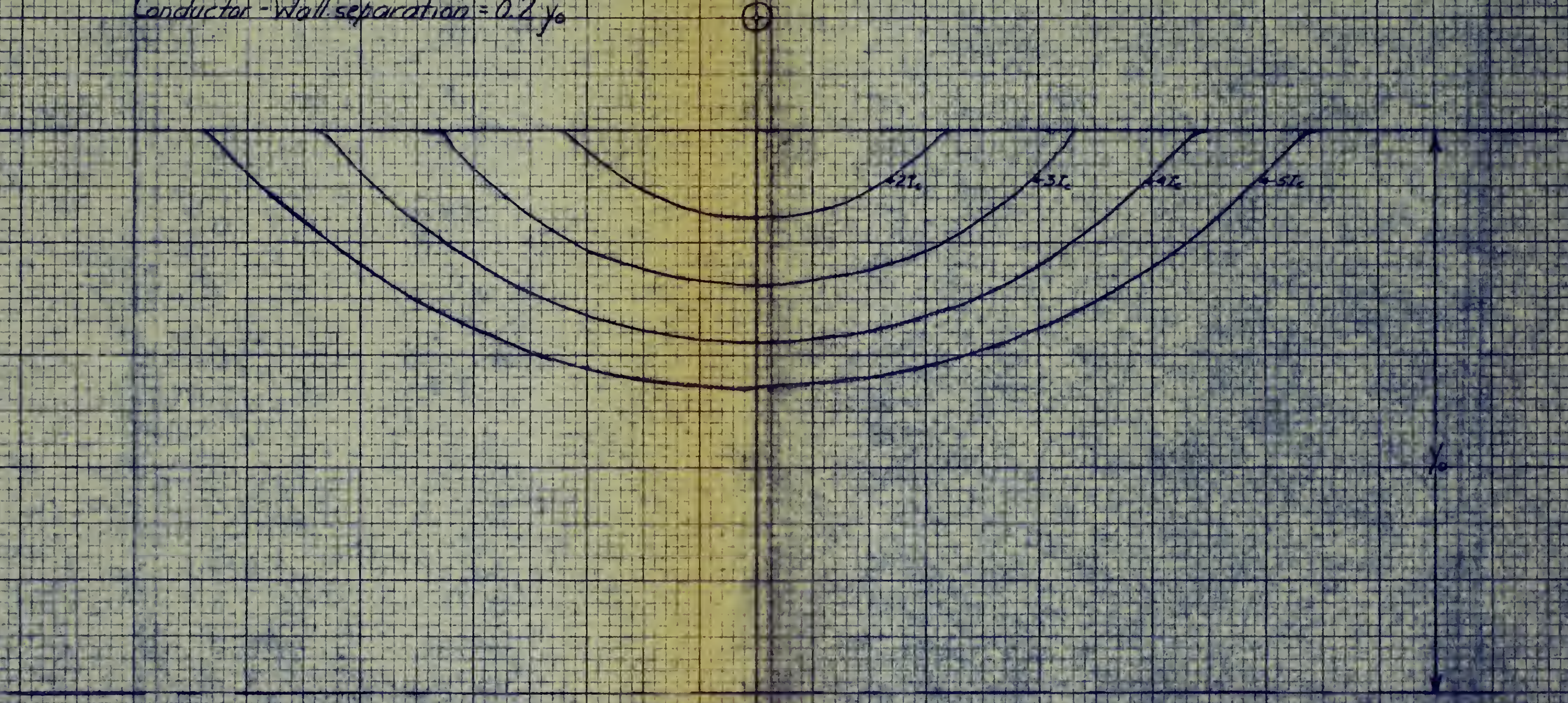


Fig 2

K+E 10 X 10 TO THE INCH KEUFFEL & ESSER CO. 358-5DL MADE IN U.S.A.

Variation of Nozzle Shape with Change in Mach Number

Constant Values:

Current = $5 I_c$

Specific Heat Ratio = 1.4

Conductor - Wall Separation = $0.2 y_0$

Note: Because of the small effect of M on nozzle shape, only one complete nozzle is shown. The throat y-coordinate for the other values of M are indicated. Note also that for values of $M > 15$, even these are indistinguishable

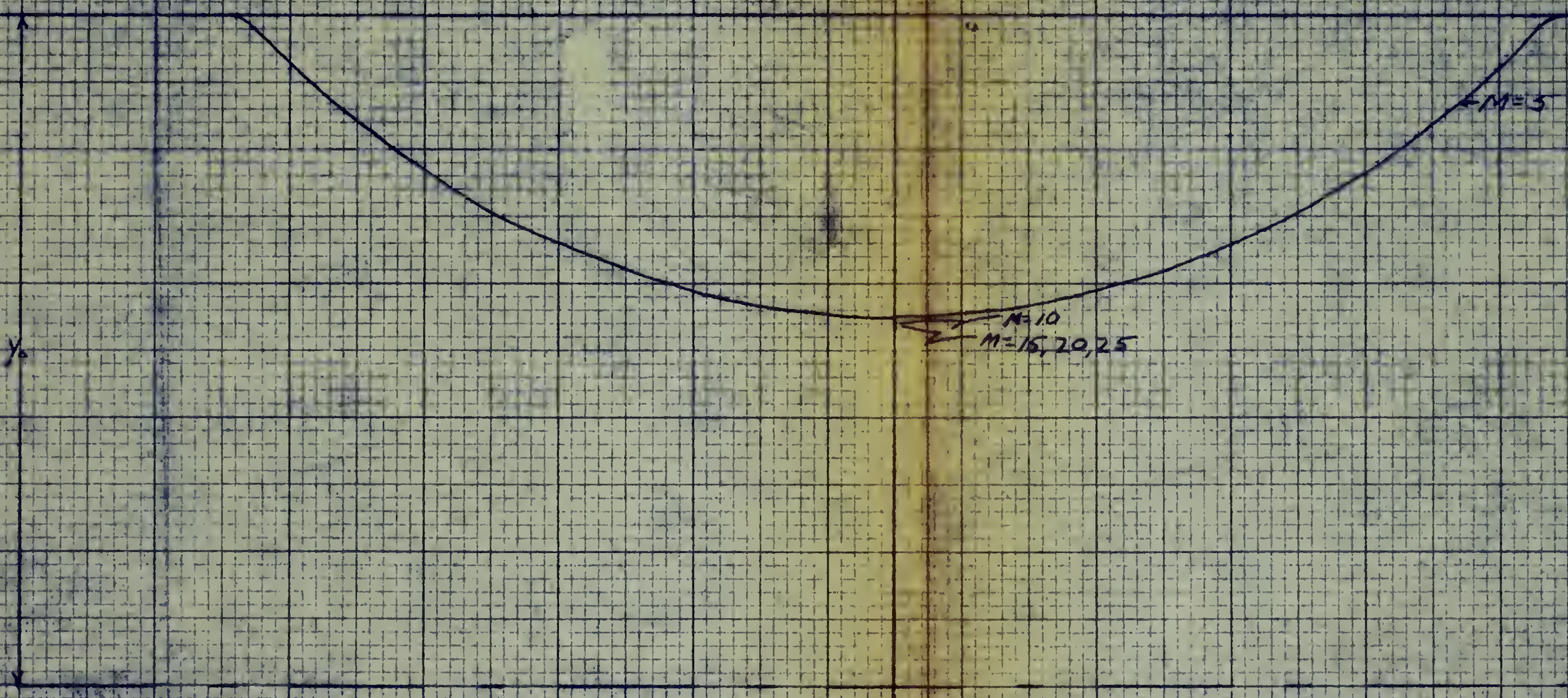


Fig. 3

Variation of Nozzle Shape with Change in Specific Heat Ratio

Constant Values:

Mach Number = 15

Current = $5 I_c$

Conductor-wall Separation = $0.2 y_0$

Note: Because of the small effect of γ on nozzle shape, only one complete nozzle is shown. The throat y-coordinate for the other values of γ are indicated

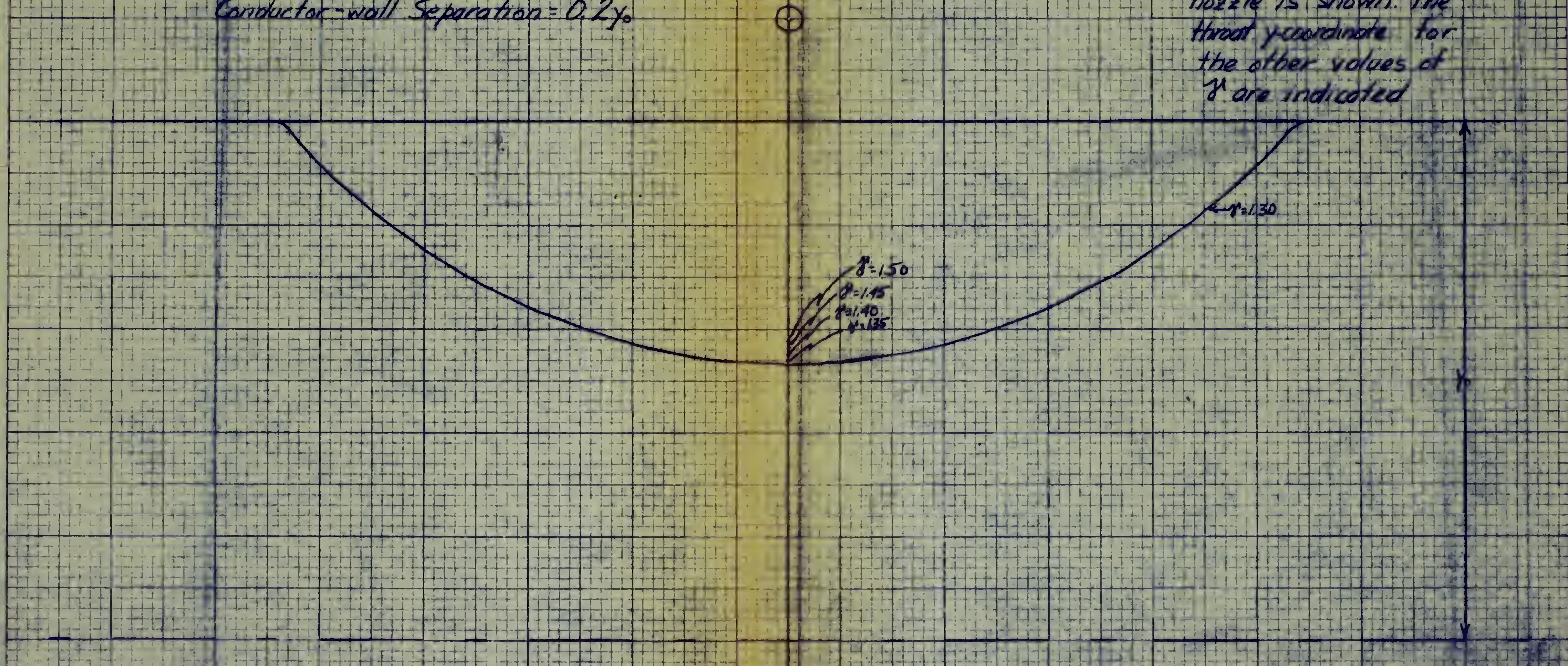


Fig 4

K+E 10 X 10 TO THE INCH 358-5DL
KEUFFEL & ESSER CO.

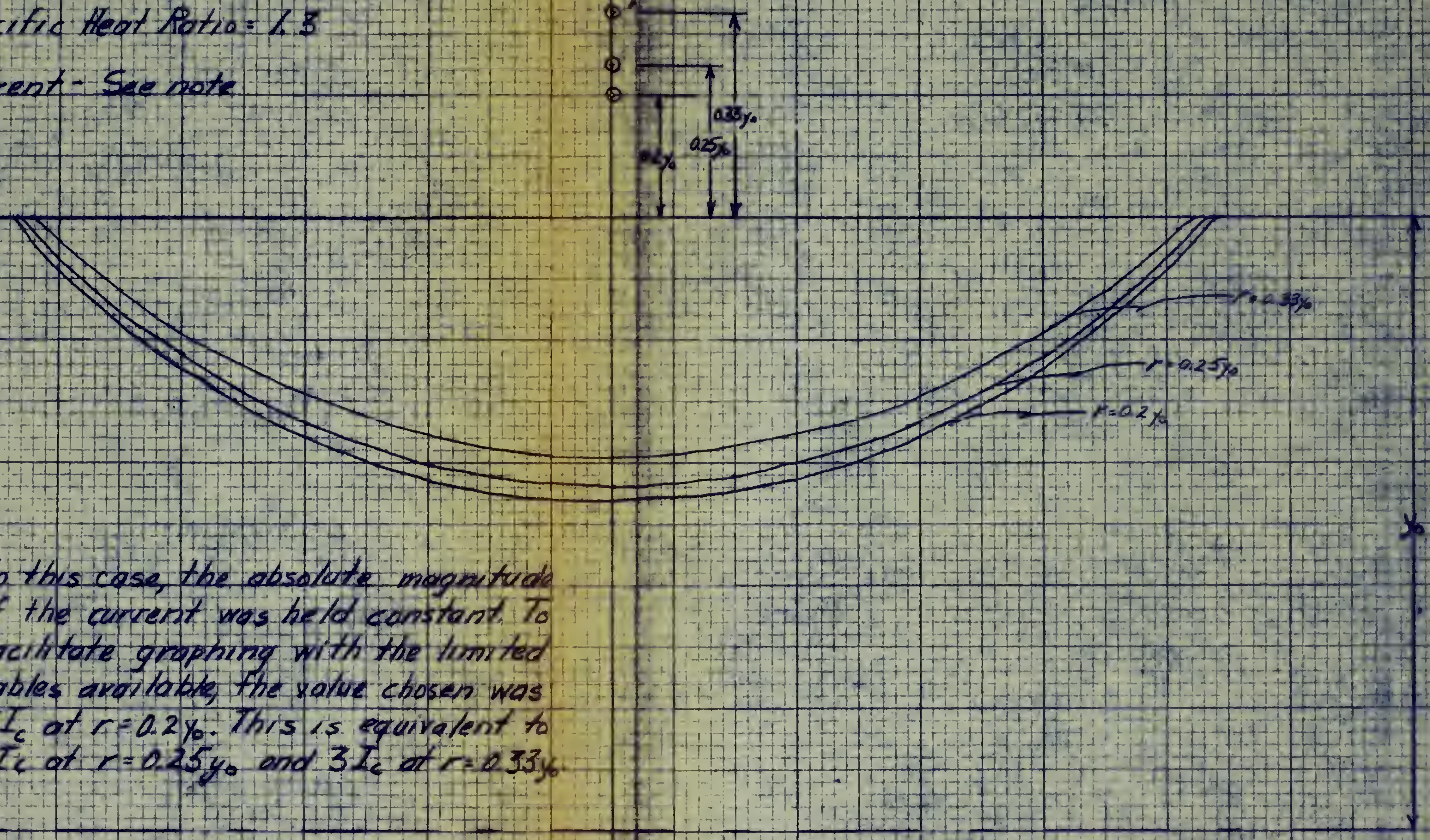
Variation in Nozzle Shape with Change in Conductor-Wall Separation

Constant Values:

Mach Number = 1.0

Specific Heat Ratio = 1.3

Current - See note



Note: In this case, the absolute magnitude of the current was held constant. To facilitate graphing with the limited tables available, the value chosen was $5 I_c$ at $r = 0.2 y_0$. This is equivalent to $4 I_c$ at $r = 0.25 y_0$ and $3 I_c$ at $r = 0.33 y_0$.

Fig. 5

K+E 10 X 10 TO THE INCH KEUFFEL & ESSER CO. 358-5DL MADE IN U.S.A.

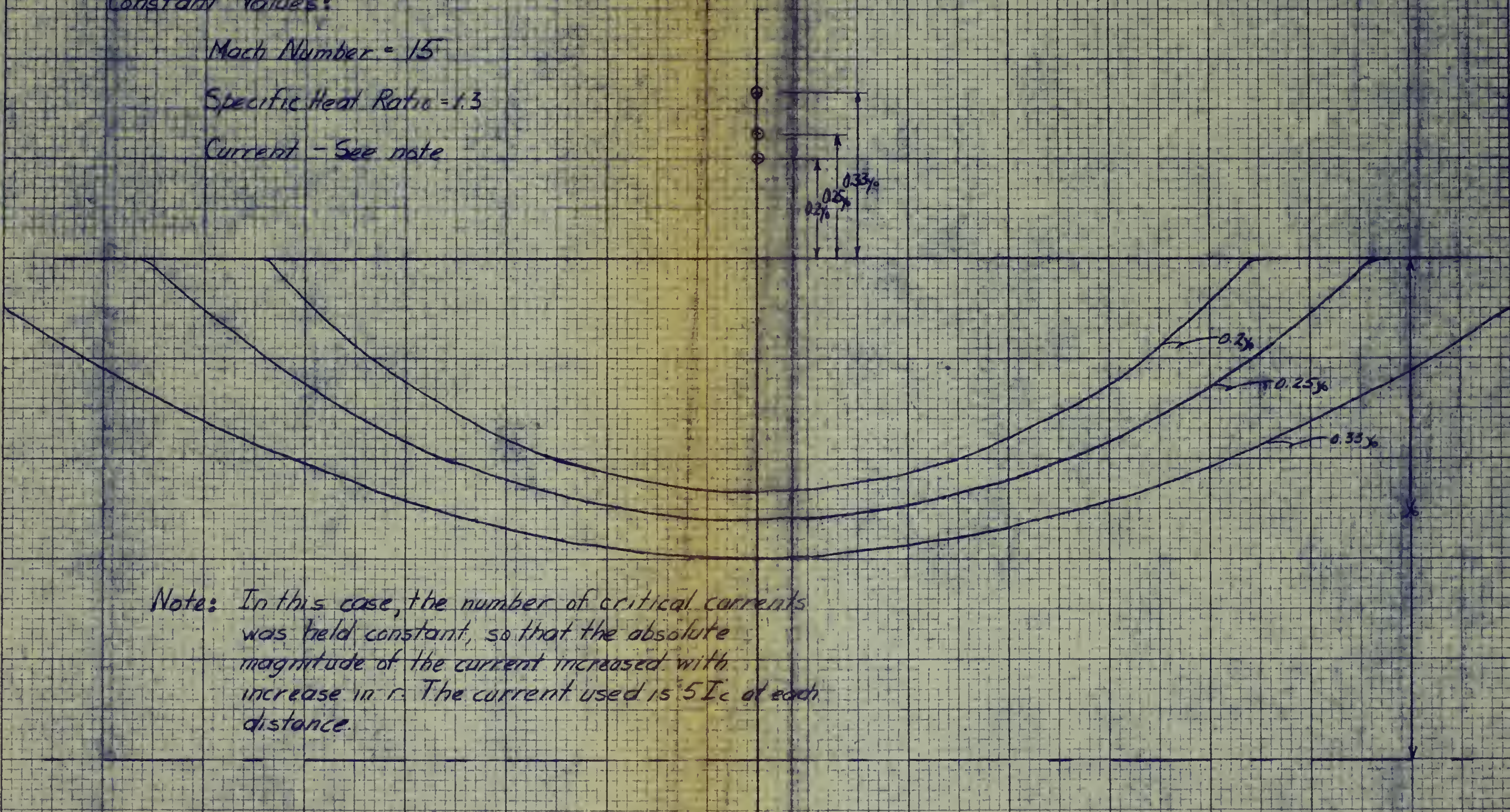
Variation in Nozzle Shape with Variation in Conductor-Wall Separation

Constant Values:

Mach Number = 15

Specific Heat Ratio = 1.3

Current - See note



Note: In this case, the number of critical currents was held constant, so that the absolute magnitude of the current increased with increase in r. The current used is $5 I_c$ at each distance.

Fig 6

K+E 10 X 10 TO THE INCH KEUFFEL & ESSER CO. 358-5DL MADE IN U.S.A.

TABLE I

f/y_o	$\gamma = 1.3$					$\gamma = 1.35$				
	M=5	M=10	M=15	M=20	M=25	M=5	M=10	M=15	M=20	M=25
.40	3.5006	3.3395	3.3135	3.3049	3.3005	3.6792	3.5000	3.4702	3.4607	3.4559
.42	3.2714	3.1317	3.1091	3.1014	3.0976	3.4291	3.2739	3.2476	3.2389	3.2346
.44	3.0689	2.9459	2.9260	2.9193	2.9116	3.2066	3.0718	3.0485	3.0408	3.0369
.46	2.8873	2.7774	2.7562	2.7541	2.7511	3.0079	2.8919	2.8707	2.8638	2.8603
.48	2.7231	2.6264	2.6090	2.6037	2.6008	2.8426	2.7275	2.7103	2.7045	2.7015
.50	2.5732	2.4911	2.4767	2.4721	2.4695	2.6777	2.5780	2.5633	2.5588	2.5561
.52	2.4395	2.3647	2.3528	2.3489	2.3466	2.5279	2.4447	2.4300	2.4259	2.4233
.54	2.3183	2.2507	2.2392	2.2353	2.2329	2.3929	2.3231	2.3109	2.3072	2.3049
.56	2.2023	2.1462	2.1367	2.1333	2.1316	2.2749	2.2075	2.1966	2.1931	2.1912
.58	2.1030	2.0494	2.0401	2.0367	2.0352	2.1633	2.1076	2.0980	2.0948	2.0929
.60	2.0040	1.9581	1.9502	1.9472	1.9462	2.0628	2.0089	1.9996	1.9970	1.9953
.62	1.9165	1.8751	1.8682	1.8656	1.8647	1.9664	1.9209	1.9130	1.9104	1.9091
.64	1.8354	1.7975	1.7916	1.7895	1.7885	1.8789	1.8399	1.8328	1.8307	1.8289
.66	1.7601	1.7271	1.7214	1.7196	1.7185	1.7982	1.7644	1.7582	1.7573	1.7552
.68	1.6893	1.6611	1.6562	1.6546	1.6536	1.7249	1.6931	1.6880	1.6861	1.6852
.70	1.6244	1.5980	1.5938	1.5924	1.5915	1.6558	1.6280	1.6233	1.6214	1.6206
.72	1.5636	1.5412	1.5371	1.5357	1.5350	1.5901	1.5668	1.5628	1.5612	1.5606
.74	1.5052	1.4861	1.4845	1.4815	1.4809	1.5302	1.5084	1.5046	1.5032	1.5026
.76	1.4528	1.4356	1.4369	1.4313	1.4308	1.4736	1.4556	1.4524	1.4514	1.4509
.78	1.4010	1.3868	1.3841	1.3834	1.3830	1.4203	1.4036	1.4008	1.4000	1.3995
.80	1.3551	1.3422	1.3399	1.3391	1.3387	1.3709	1.3574	1.3551	1.3543	1.3540
.82	1.3094	1.2979	1.2963	1.2955	1.2951	1.3239	1.3117	1.3096	1.3087	1.3084
.84	1.2683	1.2590	1.2576	1.2569	1.2566	1.2798	1.2701	1.2685	1.2677	1.2675
.86	1.2283	1.2202	1.2188	1.2183	1.2180	1.2384	1.2299	1.2285	1.2279	1.2277
.88	1.1900	1.1839	1.1828	1.1824	1.1822	1.1975	1.1911	1.1901	1.1896	1.1895
.90	1.1554	1.1503	1.1494	1.1491	1.1489	1.1617	1.1564	1.1555	1.1552	1.1551
.92	1.1209	1.1168	1.1160	1.1157	1.1156	1.1259	1.1217	1.1209	1.1207	1.1206
.94	1.0884	1.0857	1.0852	1.0849	1.0849	1.0916	1.0889	1.0884	1.0882	1.0882
.96	1.0589	1.0571	1.0568	1.0566	1.0566	1.0611	1.0592	1.0589	1.0588	1.0588
.98	1.0294	1.0285	1.0284	1.0283	1.0283	1.0305	1.0296	1.0295	1.0294	1.0294
1.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of (p/p_o) as a function of (f/y_o)

TABLE I (Continued)

f/y_o	γ = 1.4					γ = 1.45				
	M=5	M=10	M=15	M=20	M=25	M=5	M=10	M=15	M=20	M=25
.40	3.8676	3.6689	3.6331	3.6212	3.6159	4.0677	3.8420	3.8043	3.7921	3.7868
.42	3.5939	3.4227	3.3914	3.3825	3.3783	3.7714	3.5716	3.5446	3.5337	3.5289
.44	3.3544	3.2033	3.1776	3.1681	3.1650	3.4086	3.3396	3.3126	3.3027	3.2984
.46	3.1396	3.0078	2.9843	2.9773	2.9738	3.2749	3.1281	3.1044	3.0961	3.0922
.48	2.9473	2.8323	2.8096	2.8026	2.7992	3.0663	2.9386	2.9179	2.9105	2.9074
.50	2.7725	2.6751	2.6552	2.6490	2.6404	2.8804	2.7672	2.7494	2.7426	2.7382
.52	2.6157	2.5308	2.5135	2.5081	2.5057	2.7120	2.6109	2.5944	2.5888	2.5810
.54	2.4754	2.3959	2.3813	2.3767	2.3749	2.5574	2.4699	2.4545	2.4489	2.4454
.56	2.3455	2.2789	2.2643	2.2601	2.2583	2.4156	2.3413	2.3289	2.3244	2.3227
.58	2.2257	2.1674	2.1550	2.1516	2.1500	2.2928	2.2240	2.2122	2.2081	2.2064
.60	2.1195	2.0672	2.0550	2.0519	2.0500	2.1760	2.1189	2.1091	2.1056	2.1041
.62	2.0169	1.9704	1.9602	1.9575	1.9561	2.0715	2.0168	2.0076	2.0041	2.0025
.64	1.9241	1.8826	1.8742	1.8719	1.8710	1.9708	1.9248	1.9167	1.9139	1.9126
.66	1.8392	1.8016	1.7942	1.7923	1.7912	1.8801	1.8401	1.8329	1.8307	1.8295
.68	1.7604	1.7282	1.7212	1.7193	1.7182	1.7958	1.7613	1.7552	1.7532	1.7522
.70	1.6864	1.6587	1.6530	1.6513	1.6503	1.7188	1.6874	1.6824	1.6804	1.6797
.72	1.6184	1.5925	1.5881	1.5866	1.5858	1.6472	1.6194	1.6146	1.6128	1.6122
.74	1.5551	1.5326	1.5286	1.5271	1.5264	1.5795	1.5561	1.5520	1.5506	1.5500
.76	1.4943	1.4757	1.4724	1.4712	1.4706	1.5164	1.4954	1.4918	1.4908	1.4902
.78	1.4396	1.4224	1.4194	1.4183	1.4177	1.4584	1.4407	1.4375	1.4364	1.4359
.80	1.3866	1.3726	1.3701	1.3692	1.3688	1.4022	1.3876	1.3850	1.3841	1.3837
.82	1.3379	1.3253	1.3230	1.3223	1.3220	1.3521	1.3391	1.3367	1.3359	1.3356
.84	1.2906	1.2808	1.2790	1.2784	1.2782	1.3022	1.2917	1.2898	1.2892	1.2890
.86	1.2481	1.2395	1.2379	1.2374	1.2372	1.2580	1.2490	1.2473	1.2468	1.2467
.88	1.2055	1.1984	1.1971	1.1968	1.1966	1.2141	1.2064	1.2049	1.2044	1.2044
.90	1.1679	1.1625	1.1615	1.1611	1.1610	1.1741	1.1685	1.1675	1.1672	1.1672
.92	1.1310	1.1266	1.1258	1.1255	1.1254	1.1360	1.1315	1.1307	1.1305	1.1305
.94	1.0949	1.0920	1.0916	1.0913	1.0913	1.0982	1.0952	1.0948	1.0946	1.0946
.96	1.0633	1.0613	1.0611	1.0609	1.0609	1.0655	1.0635	1.0632	1.0631	1.0631
.98	1.0316	1.0307	1.0305	1.0304	1.0304	1.0327	1.0317	1.0316	1.0315	1.0315
1.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of (p/p_o) as a function of (f/y_o)

TABLE I (Continued)

$^{f}/y_o$	$\gamma = 1.5$				
	M=5	M=10	M=15	M=20	M=25
.40	4.2807	4.0254	3.9854	3.9710	3.9650
.42	3.9541	3.7373	3.7027	3.6907	3.6848
.44	3.6685	3.4824	3.4518	3.4410	3.4381
.46	3.4160	3.2549	3.2275	3.2176	3.2133
.48	3.1898	3.0500	3.0259	3.0178	3.0139
.50	2.9882	2.8673	2.8460	2.8389	2.8354
.52	2.8072	2.7020	2.6835	2.6772	2.6745
.54	2.6445	2.5507	2.5349	2.5297	2.5276
.56	2.4968	2.4118	2.3970	2.3927	2.3910
.58	2.3605	2.2894	2.2769	2.2727	2.2709
.60	2.2370	2.1730	2.1628	2.1592	2.1576
.62	2.1253	2.0691	2.0594	2.0558	2.0543
.64	2.0187	1.9691	1.9610	1.9580	1.9566
.66	1.9222	1.8789	1.8718	1.8693	1.8681
.68	1.8338	1.7950	1.7889	1.7866	1.7856
.70	1.7523	1.7187	1.7126	1.7103	1.7095
.72	1.6763	1.6472	1.6420	1.6403	1.6394
.74	1.6043	1.5796	1.5753	1.5741	1.5732
.76	1.5392	1.5169	1.5128	1.5116	1.5109
.78	1.4772	1.4590	1.4556	1.4546	1.4540
.80	1.4196	1.4029	1.3998	1.3990	1.3985
.82	1.3661	1.3525	1.3501	1.3494	1.3490
.84	1.3146	1.3025	1.3005	1.2998	1.2996
.86	1.2677	1.2584	1.2568	1.2563	1.2560
.88	1.2226	1.2147	1.2131	1.2128	1.2124
.90	1.1804	1.1746	1.1734	1.1732	1.1729
.92	1.1411	1.1365	1.1355	1.1353	1.1350
.94	1.1018	1.0985	1.0979	1.0977	1.0976
.96	1.0677	1.0657	1.0653	1.0651	1.0650
.98	1.0338	1.0328	1.0326	1.0326	1.0325
1.00	1.000	1.000	1.000	1.000	1.0000

Values of (p/p_o) as a function of (f/y_o)

TABLE II

f/y_o	M = 5				M = 10			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0689	1.6034	2.1379	2.6724	1.0945	1.6417	2.1889	2.7361
.42	1.1058	1.6586	2.2115	2.7644	1.1301	1.6952	2.2603	2.8253
.44	1.1417	1.7125	2.2834	2.8542	1.1652	1.7478	2.3305	2.9131
.46	1.1770	1.7655	2.3540	2.9426	1.2000	1.8001	2.4001	3.0001
.48	1.2120	1.8180	2.4239	3.0299	1.2341	1.8512	2.4682	3.0853
.50	1.2468	1.8702	2.4936	3.1170	1.2672	1.9008	2.5344	3.1680
.52	1.2805	1.9207	2.5610	3.2012	1.3006	1.9508	2.6011	3.2514
.54	1.3135	1.9703	2.6271	3.2839	1.3332	1.9997	2.6663	3.3329
.56	1.3477	2.0216	2.6954	3.3693	1.3652	2.0478	2.7304	3.4130
.58	1.3791	2.0687	2.7582	3.4478	1.3910	2.0956	2.7941	3.4926
.60	1.4128	2.1192	2.8257	3.5321	1.4293	2.1439	2.8586	3.5732
.62	1.4447	2.1670	2.8893	3.6117	1.4606	2.1909	2.9212	3.6515
.64	1.4762	2.2143	2.9525	3.6906	1.4918	2.2376	2.9835	3.7294
.66	1.5075	2.2612	3.0150	3.7687	1.5278	2.2828	3.0437	3.8046
.68	1.5388	2.3082	3.0776	3.8470	1.5518	2.3277	3.1037	3.8796
.70	1.5692	2.3539	3.1385	3.9231	1.5822	2.3732	3.1643	3.9554
.72	1.5995	2.3992	3.1990	3.9987	1.6111	2.4166	3.2222	4.0277
.74	1.6301	2.4452	3.2602	4.0753	1.6406	2.4608	3.2811	4.1014
.76	1.6593	2.4890	3.3187	4.1483	1.6692	2.5038	3.3383	4.1729
.78	1.6898	2.5346	3.3795	4.2244	1.6984	2.5476	3.3967	4.2459
.80	1.7181	2.5771	3.4361	4.2952	1.7264	2.5896	3.4527	4.3159
.82	1.7478	2.6217	3.4956	4.3695	1.7555	2.6332	3.5109	4.3886
.84	1.7759	2.6638	3.5518	4.4397	1.7824	2.6736	3.5647	4.4559
.86	1.8046	2.7068	3.6091	4.5114	1.8106	2.7159	3.6212	4.5265
.88	1.8333	2.7500	3.6667	4.5834	1.8381	2.7571	3.6761	4.5952
.90	1.8606	2.7910	3.7213	4.6516	1.8648	2.7972	3.7296	4.6620
.92	1.8891	2.8337	3.7782	4.7228	1.8925	2.8388	3.7850	4.7313
.94	1.9170	2.8755	3.8340	4.7925	1.9194	2.8791	3.8388	4.7985
.96	1.9436	2.9155	3.8873	4.8591	1.9451	2.9177	3.8903	4.8629
.98	1.9712	2.9568	3.9424	4.9281	1.9722	2.9583	3.9444	4.9305
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000
r		(d/r) ratio at nozzle throat						
$.20y_o$	1.7812	2.3994	2.8936	3.2911	1.7862	2.4116	2.9130	3.3181
$.25y_o$	1.7693	2.2978	2.7238	3.0560	1.7881	2.3106	2.7435	3.0826
$.33y_o$	1.6798	2.1550	2.4970	2.7496	1.6858	2.1685	2.5164	2.7743

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, γ = 1.30

TABLE II

(Continued)

f/y_o	M = 15				M = 20			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0987	1.6481	2.1974	2.7468	1.1002	1.6503	2.2003	2.7504
.42	1.1342	1.7014	2.2685	2.8356	1.1357	1.7035	2.2713	2.8391
.44	1.1692	1.7538	2.3384	2.9230	1.1705	1.7558	2.3411	2.9264
.46	1.2047	1.8070	2.4093	3.0117	1.2052	1.8078	2.4104	3.0130
.48	1.2382	1.8574	2.4765	3.0956	1.2395	1.8592	2.4789	3.0987
.50	1.2708	1.9062	2.5416	3.1770	1.2720	1.9080	2.5440	3.1801
.52	1.3039	1.9558	2.6077	3.2597	1.3050	1.9575	2.6099	3.2624
.54	1.3365	2.0048	2.6731	3.3414	1.3377	2.0066	2.6754	3.3443
.56	1.3683	2.0524	2.7365	3.4207	1.3693	2.0540	2.7386	3.4233
.58	1.4003	2.1004	2.8005	3.5007	1.4014	2.1022	2.8029	3.5036
.60	1.4322	2.1482	2.8643	3.5804	1.4333	2.1499	2.8666	3.5832
.62	1.4633	2.1949	2.9265	3.6582	1.4642	2.1964	2.9285	3.6606
.64	1.4942	2.2413	2.9884	3.7355	1.4951	2.2427	2.9902	3.7378
.66	1.5244	2.2866	3.0488	3.8110	1.5252	2.2878	3.0504	3.8130
.68	1.5541	2.3312	3.1082	3.8853	1.5548	2.3323	3.1097	3.8871
.70	1.5842	2.3762	3.1683	3.9604	1.5849	2.3774	3.1698	3.9623
.72	1.6132	2.4197	3.2263	4.0329	1.6139	2.4209	3.2279	4.0349
.74	1.6415	2.4622	3.2830	4.1037	1.6431	2.4647	3.2862	4.1078
.76	1.6685	2.5027	3.3369	4.1712	1.6717	2.5075	3.3434	4.1792
.78	1.7000	2.5499	3.3999	4.2499	1.7004	2.5506	3.4008	4.2510
.80	1.7279	2.5918	3.4557	4.3197	1.7283	2.5925	3.4566	4.3208
.82	1.7565	2.6348	3.5131	4.3914	1.7572	2.6357	3.5143	4.3929
.84	1.7835	2.6752	3.5670	4.4587	1.7840	2.6759	3.5679	4.4599
.86	1.8116	2.7174	3.6232	4.5290	1.8119	2.7179	3.6238	4.5298
.88	1.8389	2.7584	3.6778	4.5973	1.8392	2.7589	3.6785	4.5981
.90	1.8655	2.7982	3.7310	4.6637	1.8657	2.7985	3.7313	4.6642
.92	1.8932	2.8398	3.7864	4.7331	1.8934	2.8401	3.7868	4.7335
.94	1.9199	2.8799	3.8399	4.7998	1.9201	2.8802	3.8402	4.8003
.96	1.9455	2.9183	3.8911	4.8638	1.9457	2.9186	3.8914	4.8643
.98	1.9722	2.9583	3.9444	4.9305	1.9722	2.9583	3.9444	4.9305
1.00	2.000	3.000	4.000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7871	2.4137	2.9163	3.3228	1.7875	2.4146	2.9176	3.3244
$.25y_o$	1.7469	2.3129	2.7469	3.0876	1.7473	2.3136	2.7481	3.0890
$.33y_o$	1.6862	2.1709	2.5200	2.7785	1.6873	2.1718	2.5211	2.7800

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, γ = 1.30

TABLE II

(Continued)

f/y_o	M = 25			
	2Ic	3Ic	4Ic	5Ic
.40	1.1009	1.6513	2.2018	2.7522
.42	1.1364	1.7045	2.2727	2.8409
.44	1.1721	1.7582	2.3443	2.9303
.46	1.2058	1.8088	2.4117	3.0146
.48	1.2402	1.8602	2.4803	3.1004
.50	1.2727	1.9090	2.5453	3.1817
.52	1.3056	1.9584	2.6114	3.2639
.54	1.3384	2.0076	2.6768	3.3460
.56	1.3699	2.0548	2.7397	3.4247
.58	1.4019	2.1029	2.8039	3.5048
.60	1.4336	2.1504	2.8672	3.5840
.62	1.4647	2.1970	2.9293	3.6617
.64	1.4956	2.2433	2.9911	3.7389
.66	1.5257	2.2885	3.0513	3.8142
.68	1.5553	2.3330	3.1107	3.8833
.70	1.5854	2.3781	3.1708	3.9635
.72	1.6143	2.4215	3.2287	4.0358
.74	1.6435	2.4653	3.2870	4.1088
.76	1.6720	2.5079	3.3439	4.1799
.78	1.7007	2.5510	3.4014	4.2517
.80	1.7286	2.5929	3.4572	4.3215
.82	1.7575	2.6362	3.5149	4.3937
.84	1.7841	2.6762	3.5682	4.4603
.86	1.8123	2.7184	3.6245	4.5306
.88	1.8394	2.7591	3.6788	4.5985
.90	1.8658	2.7988	3.7317	4.6646
.92	1.8936	2.8404	3.7872	4.7340
.94	1.9201	2.8802	3.8402	4.8003
.96	1.9457	2.9186	3.8914	4.8643
.98	1.9722	2.9583	3.9444	4.9305
1.00	2.0000	3.000	4.000	5.0000

r	(d/r) ratio at nozzle throat			
$.20y_o$	1.7876	2.4150	2.9181	3.3253
$.25y_o$	1.7475	2.3141	2.7487	3.0898
$.33y_o$	1.6875	2.1721	2.5217	2.7807

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.30$

TABLE III

f/y_o	M = 5				M = 10			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0427	1.5640	2.0854	2.6067	1.0691	1.6036	2.1381	2.6727
.42	1.0800	1.6200	2.1601	2.7001	1.1053	1.6580	2.2107	2.7633
.44	1.1169	1.6753	2.2338	2.7922	1.1411	1.7116	2.2822	2.8527
.46	1.1532	1.7298	2.3064	2.8830	1.1761	1.7641	2.3521	2.9401
.48	1.1862	1.7794	2.3725	2.9656	1.2110	1.8165	2.4220	3.0276
.50	1.2222	1.8333	2.4444	3.0555	1.2456	1.8685	2.4913	3.1141
.52	1.2579	1.8870	2.5159	3.1449	1.2791	1.9186	2.5582	3.1977
.54	1.2929	1.9394	2.5859	3.2323	1.3122	1.9682	2.6243	3.2804
.56	1.3260	1.9890	2.6520	3.3150	1.3461	2.0191	2.6922	3.3652
.58	1.3598	2.0397	2.7196	3.3995	1.3776	2.0664	2.7552	3.4440
.60	1.3926	2.0888	2.7851	3.4814	1.4111	2.1167	2.8223	3.5278
.62	1.4262	2.1393	2.8525	3.5656	1.4430	2.1645	2.8860	3.6075
.64	1.4591	2.1887	2.9182	3.6478	1.4750	2.2117	2.9490	3.6862
.66	1.4914	2.2371	2.9828	3.7286	1.5057	2.2585	3.0114	3.7642
.68	1.5228	2.2841	3.0455	3.8069	1.5370	2.3056	3.0741	3.8426
.70	1.5542	2.3314	3.1085	3.8856	1.5675	2.3513	3.1350	3.9188
.72	1.5860	2.3791	3.1721	3.9651	1.5978	2.3967	3.1957	3.9946
.74	1.6168	2.4252	3.2336	4.0420	1.6284	2.4426	3.2568	4.0710
.76	1.6476	2.4714	3.2952	4.1189	1.6577	2.4865	3.3154	4.1442
.78	1.6781	2.5172	3.3563	4.1953	1.6882	2.5323	3.3764	4.2205
.80	1.7081	2.5621	3.4162	4.2702	1.7166	2.5749	3.4332	4.2915
.82	1.7382	2.6073	3.4764	4.3456	1.7463	2.6194	3.4925	4.3657
.84	1.7679	2.6518	3.5358	4.4197	1.7746	2.6619	3.5492	4.4366
.86	1.7973	2.6959	3.5945	4.4932	1.8034	2.7051	3.6069	4.5086
.88	1.8277	2.7415	3.6553	4.5691	1.8325	2.7488	3.6650	4.5813
.90	1.8556	2.7834	3.7113	4.6391	1.8598	2.7897	3.7195	4.6494
.92	1.8848	2.8272	3.7697	4.7121	1.8884	2.8326	3.7768	4.7210
.94	1.9142	2.8714	3.8285	4.7856	1.9166	2.8749	3.8333	4.7916
.96	1.9416	2.9123	3.8831	4.8539	1.9433	2.9149	3.8865	4.8581
.98	1.9702	2.9554	3.9405	4.9256	1.9710	2.9565	3.9421	4.9276
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7752	2.3857	2.8713	3.2629	1.7803	2.3977	2.8914	3.2894

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.35$

TABLE III
(Continued)

f/y_o	M = 15				M = 20			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0737	1.6105	2.1473	2.6841	1.0751	1.6126	2.1502	2.6877
.42	1.1098	1.6647	2.2196	2.7745	1.1113	1.6669	2.2226	2.7782
.44	1.1455	1.7182	2.2910	2.8637	1.1469	1.7204	2.2938	2.8673
.46	1.1804	1.7706	2.3609	2.9511	1.1818	1.7727	2.3636	2.9546
.48	1.2148	1.8223	2.4297	3.0371	1.2162	1.8243	2.4324	3.0404
.50	1.2492	1.8738	2.4984	3.1230	1.2503	1.8755	2.5006	3.1258
.52	1.2830	1.9246	2.5661	3.2076	1.2841	1.9262	2.5682	3.2103
.54	1.3156	1.9734	2.6312	3.2890	1.3167	1.9751	2.6335	3.2919
.56	1.3494	2.0242	2.6989	3.3736	1.3505	2.0258	2.7011	3.3763
.58	1.3808	2.0713	2.7617	3.4521	1.3819	2.0728	2.7638	3.4547
.60	1.4143	2.1215	2.8287	3.5358	1.4152	2.1228	2.8305	3.5381
.62	1.4460	2.1690	2.8921	3.6151	1.4470	2.1705	2.8939	3.6174
.64	1.4773	2.2160	2.9546	3.6933	1.4782	2.2173	2.9564	3.6955
.66	1.5083	2.2624	3.0166	3.7707	1.5088	2.2631	3.0175	3.7719
.68	1.5394	2.3091	3.0788	3.8485	1.5402	2.3104	3.0805	3.8506
.70	1.5697	2.3546	3.1395	3.9243	1.5707	2.3561	3.1414	3.9268
.72	1.5999	2.3998	3.1997	3.9997	1.6006	2.4010	3.2013	4.0016
.74	1.6305	2.4458	3.2610	4.0763	1.6312	2.4468	3.2624	4.0780
.76	1.6595	2.4892	3.3189	4.1487	1.6602	2.4902	3.3203	4.1504
.78	1.6899	2.5349	3.3798	4.2248	1.6903	2.5355	3.3807	4.2258
.80	1.7181	2.5771	3.4361	4.2952	1.7187	2.5780	3.4373	4.2966
.82	1.7476	2.6215	3.4953	4.3691	1.7483	2.6224	3.4965	4.3706
.84	1.7757	2.6636	3.5515	4.4393	1.7764	2.6645	3.5527	4.4409
.86	1.8044	2.7066	3.6088	4.5110	1.8049	2.7073	3.6098	4.5122
.88	1.8333	2.7500	3.6667	4.5834	1.8337	2.7505	3.6674	4.5842
.90	1.8606	2.7910	3.7213	4.6516	1.8608	2.7912	3.7216	4.6520
.92	1.8891	2.8337	3.7782	4.7228	1.8893	2.8339	3.7786	4.7232
.94	1.9170	2.8755	3.8340	4.7925	1.9172	2.8758	3.8344	4.7929
.96	1.9436	2.9155	3.8873	4.8591	1.9436	2.9154	3.8873	4.8591
.98	1.9712	2.9568	3.9424	4.9281	1.9712	2.9568	3.9424	4.9281
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7811	2.3999	2.8951	3.2920	1.7816	2.4007	2.8962	3.2956

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, γ = 1.35

TABLE III
(Continued)

f/y_o	M = 25			
	2Ic	3Ic	4Ic	5Ic
.40	1.0758	1.6138	2.1517	2.6896
.42	1.1120	1.6681	2.2241	2.7801
.44	1.1476	1.7215	2.2953	2.8691
.46	1.1826	1.7739	2.3652	2.9565
.48	1.2168	1.8253	2.4337	3.0421
.50	1.2509	1.8764	2.5019	3.1273
.52	1.2848	1.9272	2.5695	3.2119
.54	1.3173	1.9760	2.6347	3.2934
.56	1.3511	2.0266	2.7022	3.3777
.58	1.3825	2.0737	2.7649	3.4561
.60	1.4158	2.1237	2.8317	3.5396
.62	1.4475	2.1712	2.8950	3.6187
.64	1.4789	2.2183	2.9577	3.6971
.66	1.5097	2.2645	3.0193	3.7741
.68	1.5406	2.3109	3.0812	3.8515
.70	1.5711	2.3566	3.1422	3.9277
.72	1.6010	2.4015	3.2020	4.0026
.74	1.6316	2.4474	3.2632	4.0790
.76	1.6604	2.4907	3.3209	4.1511
.78	1.6906	2.5359	3.3812	4.2265
.80	1.7188	2.5782	3.4376	4.2970
.82	1.7486	2.6228	3.4971	4.3714
.84	1.7765	2.6648	3.5530	4.4413
.86	1.8051	2.7076	3.6101	4.5127
.88	1.8339	2.7508	3.6677	4.5846
.90	1.8608	2.7912	3.7216	4.6520
.92	1.8893	2.8340	3.7786	4.7232
.94	1.9172	2.8758	3.8344	4.7929
.96	1.9436	2.9155	3.8873	4.8591
.98	1.9712	2.9568	3.9424	4.9281
1.00	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat			
$.20y_o$	1.7817	2.4010	2.8969	3.2964

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.35$

TABLE IV

$^{f}/y_o$	M = 5				M = 10			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0170	1.5255	2.0340	2.5425	1.0442	1.5663	2.0883	2.6104
.42	1.0550	1.5824	2.1099	2.6374	1.0810	1.6215	2.1620	2.7026
.44	1.0920	1.6380	2.1840	2.7300	1.1174	1.6762	2.2349	2.7936
.46	1.1287	1.6931	2.2575	2.8218	1.1532	1.7298	2.3064	2.8830
.48	1.1650	1.7474	2.3299	2.9124	1.1884	1.7826	2.3769	2.9711
.50	1.2011	1.8017	2.4023	3.0028	1.2228	1.8342	2.4456	3.0570
.52	1.2366	1.8549	2.4733	3.0916	1.2572	1.8858	2.5145	3.1431
.54	1.2712	1.9068	2.5424	3.1780	1.2921	1.9381	2.5841	3.2302
.56	1.3059	1.9589	2.6118	3.2648	1.3249	1.9873	2.6497	3.3121
.58	1.3406	2.0109	2.6811	3.3514	1.3585	2.0378	2.7170	3.3963
.60	1.3737	2.0606	2.7474	3.4343	1.3910	2.0865	2.7820	3.4775
.62	1.4083	2.1124	2.8165	3.5206	1.4248	2.1372	2.8496	3.5620
.64	1.4419	2.1628	2.8837	3.6046	1.4576	2.1864	2.9152	3.6440
.66	1.4747	2.2121	2.9494	3.6868	1.4880	2.2320	2.9760	3.7200
.68	1.5074	2.2611	3.0148	3.7685	1.5214	2.2821	3.0428	3.8034
.70	1.5401	2.3102	3.0802	3.8503	1.5529	2.3294	3.1058	3.8823
.72	1.5721	2.3581	3.1442	3.9302	1.5849	2.3774	3.1698	3.9623
.74	1.6038	2.4058	3.2077	4.0096	1.6155	2.4233	3.2310	4.0388
.76	1.6361	2.4542	3.2723	4.0903	1.6464	2.4695	3.2927	4.1159
.78	1.6669	2.5004	3.3339	4.1674	1.6770	2.5155	3.3540	4.1925
.80	1.6985	2.5478	3.3970	4.2463	1.7071	2.5606	3.4141	4.2677
.82	1.7291	2.5936	3.4581	4.3226	1.7373	2.6060	3.4746	4.3433
.84	1.7606	2.6408	3.5211	4.4014	1.7673	2.6509	3.5345	4.4181
.86	1.7902	2.6853	3.5804	4.4755	1.7965	2.6947	3.5929	4.4912
.88	1.8215	2.7322	3.6430	4.5537	1.8270	2.7405	3.6540	4.5675
.90	1.8507	2.7760	3.7013	4.6266	1.8549	2.7824	3.7099	4.6374
.92	1.8806	2.8209	3.7612	4.7015	1.8843	2.8265	3.7686	4.7108
.94	1.9113	2.8670	3.8226	4.7783	1.9139	2.8708	3.8278	4.7847
.96	1.9395	2.9092	3.8790	4.8487	1.9414	2.9121	3.8827	4.8534
.98	1.9691	2.9536	3.9382	4.9227	1.9701	2.9551	3.9401	4.9251
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7696	2.3716	2.8501	3.2347	1.7747	2.3845	2.8696	3.2616

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.40$

TABLE IV
(Continued)

f/y_o	M = 15				M = 20			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0493	1.5739	2.0985	2.6232	1.0510	1.5765	2.1021	2.6276
.42	1.0860	1.6290	2.1720	2.7150	1.0874	1.6311	2.1749	2.7186
.44	1.1220	1.6829	2.2439	2.8049	1.1237	1.6855	2.2473	2.8091
.46	1.1577	1.7366	2.3155	2.8944	1.1591	1.7386	2.3182	2.8977
.48	1.1932	1.7898	2.3864	2.9829	1.1947	1.7920	2.3893	2.9867
.50	1.2274	1.8411	2.4547	3.0684	1.2288	1.8432	2.4576	3.0720
.52	1.2615	1.8923	2.5230	3.1538	1.2628	1.8942	2.5256	3.1570
.54	1.2961	1.9441	2.5922	3.2402	1.2973	1.9469	2.5945	3.2432
.56	1.3291	1.9936	2.6582	3.3227	1.3303	1.9955	2.6606	3.3258
.58	1.3624	2.0436	2.7248	3.4060	1.3635	2.0453	2.7270	3.4088
.60	1.3952	2.0928	2.7904	3.4880	1.3963	2.0944	2.7925	3.4906
.62	1.4285	2.1427	2.8569	3.5712	1.4295	2.1442	2.8590	3.5737
.64	1.4609	2.1914	2.9218	3.6523	1.4618	2.1927	2.9235	3.6544
.66	1.4931	2.2396	2.9862	3.7327	1.4939	2.2408	2.9878	3.7347
.68	1.5245	2.2868	3.0490	3.8113	1.5253	2.2880	3.0506	3.8133
.70	1.5555	2.3332	3.1109	3.8886	1.5564	2.3346	3.1128	3.8911
.72	1.5870	2.3806	3.1741	3.9676	1.5878	2.3817	3.1756	3.9695
.74	1.6176	2.4264	3.2352	4.0440	1.6184	2.4276	3.2368	4.0460
.76	1.6483	2.4724	3.2965	4.1207	1.6489	2.4734	3.2979	4.1224
.78	1.6787	2.5180	3.3574	4.1967	1.6794	2.5191	3.3588	4.1985
.80	1.7087	2.5630	3.4173	4.2717	1.7093	2.5639	3.4185	4.2731
.82	1.7388	2.6082	3.4777	4.3471	1.7393	2.6089	3.4786	4.3482
.84	1.7685	2.6528	3.5370	4.4213	1.7688	2.6532	3.5376	4.4220
.86	1.7976	2.6964	3.5952	4.4940	1.7979	2.6969	3.5958	4.4948
.88	1.8280	2.7420	3.6560	4.5700	1.8282	2.7422	3.6563	4.5704
.90	1.8558	2.7837	3.7116	4.6395	1.8561	2.7842	3.7123	4.6404
.92	1.8850	2.8275	3.7700	4.7125	1.8852	2.8278	3.7704	4.7130
.94	1.9142	2.8714	3.8285	4.7856	1.9144	2.8716	3.8289	4.7861
.96	1.9416	2.9123	3.8831	4.8539	1.9417	2.9126	3.8835	4.8544
.98	1.9702	2.9554	3.9405	4.9256	1.9702	2.9554	3.9405	4.9256
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7756	2.3867	2.8739	3.2672	1.7758	2.3875	2.8751	3.2689

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.40$

TABLE IV
(Continued)

f/y_o	M = 25·			
	2Ic	3Ic	4Ic	5Ic
.40	1.0517	1.5776	2.1035	2.6294
.42	1.0881	1.6322	2.1763	2.7203
.44	1.1242	1.6863	2.2485	2.8106
.46	1.1598	1.7396	2.3195	2.8994
.48	1.1954	1.7931	2.3908	2.9885
.50	1.2308	1.8463	2.4617	3.0771
.52	1.2635	1.8953	2.5270	3.1588
.54	1.2978	1.9467	2.5955	3.2444
.56	1.3308	1.9963	2.6617	3.3271
.58	1.3640	2.0460	2.7280	3.4099
.60	1.3968	2.0953	2.7937	3.4921
.62	1.4300	2.1450	2.8600	3.5750
.64	1.4622	2.1933	2.9244	3.6555
.66	1.4943	2.2415	2.9886	3.7358
.68	1.5258	2.2887	3.0516	3.8145
.70	1.5569	2.3354	3.1138	3.8923
.72	1.5882	2.3823	3.1764	3.9705
.74	1.6188	2.4282	3.2376	4.0469
.76	1.6492	2.4738	3.2984	4.1230
.78	1.6797	2.5195	3.3594	4.1992
.80	1.7094	2.5641	3.4188	4.2735
.82	1.7394	2.6091	3.4789	4.3486
.84	1.7690	2.6535	3.5379	4.4224
.86	1.7981	2.6971	3.5962	4.4952
.88	1.8283	2.7425	3.6566	4.5708
.90	1.8561	2.7842	3.7123	4.6404
.92	1.8854	2.8281	3.7707	4.7134
.94	1.9144	2.8716	3.8289	4.7861
.96	1.9417	2.9126	3.8835	4.8544
.98	1.9702	2.9554	3.9405	4.9256
1.00	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat			
$.20y_o$	1.7760	2.3879	2.8757	3.2696

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.40$

TABLE V

f/y_o	M = 5				M = 10			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	0.9916	1.4874	1.9832	2.4791	1.0204	1.5305	2.0407	2.5509
.42	1.0299	1.5448	2.0597	2.5747	1.0583	1.5874	2.1165	2.6456
.44	1.0677	1.6016	2.1355	2.6694	1.0944	1.6416	2.1888	2.7360
.46	1.1052	1.6577	2.2103	2.7629	1.1308	1.6963	2.2617	2.8271
.48	1.1421	1.7132	2.2843	2.8553	1.1667	1.7501	2.3335	2.9168
.50	1.1784	1.7676	2.3568	2.9460	1.2023	1.8034	2.4046	3.0057
.52	1.2145	1.8217	2.4290	3.0362	1.2378	1.8567	2.4756	3.0944
.54	1.2506	1.8759	2.5013	3.1266	1.2726	1.9089	2.5452	3.1815
.56	1.2868	1.9303	2.5737	3.2171	1.3071	1.9607	2.6142	3.2678
.58	1.3208	1.9812	2.6417	3.3021	1.3411	2.0117	2.6822	3.3528
.60	1.3558	2.0338	2.7117	3.3896	1.3740	2.0610	2.7480	3.4350
.62	1.3896	2.0843	2.7791	3.4739	1.4084	2.1125	2.8167	3.5209
.64	1.4246	2.1369	2.8492	3.5615	1.4415	2.1623	2.8831	3.6039
.66	1.4586	2.1879	2.9172	3.6464	1.4744	2.2116	2.9488	3.6860
.68	1.4924	2.2386	2.9849	3.7311	1.5070	2.2606	3.0141	3.7676
.70	1.5255	2.2883	3.0511	3.8139	1.5396	2.3095	3.0793	3.8491
.72	1.5584	2.3375	3.1167	3.8959	1.5716	2.3574	3.1432	3.9290
.74	1.5913	2.3870	3.1827	3.9784	1.6033	2.4050	3.2067	4.0083
.76	1.6242	2.4363	3.2483	4.0604	1.6355	2.4532	3.7709	4.0886
.78	1.6562	2.4843	3.3124	4.1404	1.6662	2.4994	3.3325	4.1656
.80	1.6890	2.5336	3.3781	4.2226	1.6978	2.5467	3.3956	4.2445
.82	1.7200	2.5800	3.4398	4.3000	1.7283	2.5925	3.4566	4.3208
.84	1.7527	2.6290	3.5054	4.3817	1.7598	2.6397	3.5196	4.3995
.86	1.7832	2.6747	3.5663	4.4579	1.7895	2.6843	3.5791	4.4739
.88	1.8150	2.7226	3.6301	4.5376	1.8208	2.7312	3.6417	4.5521
.90	1.8457	2.7685	3.6914	4.6142	1.8501	2.7752	3.7003	4.6253
.92	1.8765	2.8148	3.7530	4.6913	1.8802	2.8203	3.7605	4.7006
.94	1.9084	2.8626	3.8168	4.7710	1.9111	2.8667	3.8223	4.7778
.96	1.9376	2.9064	3.8752	4.8440	1.9393	2.9089	3.8786	4.8482
.98	1.9681	2.9522	3.9362	4.9203	1.9691	2.9536	3.9382	4.9227
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7637	2.3582	2.8289	3.2090	1.7690	2.3711	2.8499	3.2364

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.45$

TABLE V
(Continued)

f/y_o	M = 15				M = 20			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0254	1.5381	2.0508	2.5634	1.0271	1.5406	2.0541	2.5677
.42	1.0623	1.5935	2.1246	2.6558	1.0639	1.5959	2.1279	2.6599
.44	1.0988	1.6483	2.1977	2.7471	1.1005	1.6508	2.2011	2.7513
.46	1.1351	1.7027	2.2703	2.8378	1.1366	1.7049	2.2732	2.8416
.48	1.1708	1.7562	2.3416	2.9271	1.1723	1.7585	2.3447	2.9308
.50	1.2062	1.8093	2.4124	3.0155	1.2076	1.8115	2.4153	3.0191
.52	1.2417	1.8625	2.4834	3.1042	1.2430	1.8645	2.4860	3.1075
.54	1.2766	1.9149	2.5531	3.1914	1.2780	1.9171	2.5561	3.1951
.56	1.3105	1.9658	2.6211	3.2763	1.3118	1.9677	2.6236	3.2795
.58	1.3448	2.0171	2.6894	3.3618	1.3459	2.0188	2.6918	3.3647
.60	1.3771	2.0657	2.7543	3.4428	1.3783	2.0674	2.7565	3.4457
.62	1.4115	2.1173	2.8231	3.5288	1.4127	2.1191	2.8255	3.5318
.64	1.4447	2.1670	2.8893	3.6117	1.4457	2.1686	2.8914	3.6143
.66	1.4773	2.2160	2.9546	3.6933	1.4782	2.2173	2.9564	3.6955
.68	1.5097	2.2645	3.0193	3.7742	1.5105	2.2657	3.0209	3.7761
.70	1.5419	2.3129	3.0838	3.8548	1.5429	2.3143	3.0857	3.8571
.72	1.5739	2.3610	3.1479	3.9348	1.5748	2.3622	3.1496	3.9370
.74	1.6054	2.4081	3.2108	4.0135	1.6062	2.4093	3.2123	4.0154
.76	1.6375	2.4562	3.2749	4.0937	1.6380	2.4570	3.2760	4.0950
.78	1.6681	2.5021	3.3361	4.1701	1.6688	2.5031	3.3375	4.1719
.80	1.6994	2.5491	3.3988	4.2484	1.7000	2.5499	3.3999	4.2499
.82	1.7298	2.5947	3.4596	4.3245	1.7304	2.5956	3.4608	4.3260
.84	1.7610	2.6415	3.5221	4.4026	1.7615	2.6422	3.5230	4.4037
.86	1.7908	2.6862	3.5817	4.4771	1.7912	2.6867	3.5823	4.4779
.88	1.8220	2.7330	3.6440	4.5550	1.8223	2.7335	3.6446	4.5558
.90	1.8510	2.7765	3.7020	4.6275	1.8512	2.7767	3.7023	4.6279
.92	1.8810	2.8214	3.7619	4.7023	1.8811	2.8217	3.7622	4.7028
.94	1.9115	2.8672	3.8230	4.7787	1.9117	2.8675	3.8234	4.7792
.96	1.9397	2.9095	3.8794	4.8492	1.9397	2.9095	3.8794	4.8492
.98	1.9691	2.9536	3.9382	4.9227	1.9693	2.9539	3.9386	4.9232
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7700	2.3735	2.8537	3.2413	1.7703	2.3743	2.8551	3.2431

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, γ = 1.45

TABLE V
(Continued)

f/y_o	M = 25			
	2Ic	3Ic	4Ic	5Ic
.40	1.0277	1.5416	2.0555	2.5694
.42	1.0647	1.5970	2.1294	2.6617
.44	1.1013	1.6519	2.2025	2.7532
.46	1.1373	1.7060	2.2747	2.8433
.48	1.1730	1.7594	2.3459	2.9324
.50	1.2086	1.8129	2.4172	3.0215
.52	1.2449	1.8674	2.4899	3.1124
.54	1.2789	1.9184	2.5579	3.1973
.56	1.3123	1.9685	2.6247	3.2808
.58	1.3464	2.0197	2.6929	3.3661
.60	1.3787	2.0681	2.7575	3.4468
.62	1.4133	2.1200	2.8267	3.5333
.64	1.4461	2.1692	2.8923	3.6153
.66	1.4786	2.2180	2.9573	3.6966
.68	1.5109	2.2664	3.0218	3.7773
.70	1.5432	2.3148	3.0864	3.8580
.72	1.5752	2.3628	3.1504	3.9379
.74	1.6064	2.4096	3.2129	4.0161
.76	1.6384	2.4576	3.2768	4.0960
.78	1.6690	2.5035	3.3381	4.1726
.80	1.7002	2.5504	3.4005	4.2506
.82	1.7306	2.5958	3.4611	4.3264
.84	1.7616	2.6425	3.5233	4.4041
.86	1.7912	2.6867	3.5823	4.4779
.88	1.8223	2.7335	3.6446	4.5558
.90	1.8512	2.7767	3.7023	4.6279
.92	1.8811	2.8217	3.7622	4.7028
.94	1.9117	2.8675	3.8234	4.7792
.96	1.9397	2.9095	3.8794	4.8492
.98	1.9693	2.9539	3.9386	4.9232
1.00	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat			
$.20y_o$	1.7704	2.3747	2.8557	3.2440

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.45$

TABLE VI

f/y_o	M = 5				M = 10			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	0.9667	1.4500	1.9333	2.4166	0.9969	1.4953	1.9937	2.4921
.42	1.0058	1.5087	2.0116	2.5145	1.0346	1.5518	2.0691	2.5864
.44	1.0442	1.5663	2.0884	2.6106	1.0718	1.6076	2.1435	2.6794
.46	1.0821	1.6232	2.1643	2.7053	1.1086	1.6629	2.2172	2.7715
.48	1.1198	1.6797	2.2396	2.8000	1.1452	1.7178	2.2904	2.8630
.50	1.1570	1.7355	2.3140	2.8925	1.1811	1.7717	2.3623	2.9528
.52	1.1937	1.7905	2.3873	2.9842	1.2167	1.8250	2.4334	3.0417
.54	1.2299	1.8448	2.4597	3.0747	1.2523	1.8784	2.5045	3.1307
.56	1.2657	1.8986	2.5315	3.1644	1.2878	1.9317	2.5757	3.2196
.58	1.3017	1.9526	2.6035	3.2544	1.3218	1.9827	2.6436	3.3045
.60	1.3372	2.0057	2.6743	3.3429	1.3568	2.0351	2.7135	3.3919
.62	1.3719	2.0579	2.7439	3.4298	1.3904	2.0857	2.7809	3.4761
.64	1.4077	2.1115	2.8153	3.5191	1.4253	2.1380	2.8506	3.5633
.66	1.4426	2.1639	2.8852	3.6065	1.4591	2.1887	2.9182	3.6478
.68	1.4769	2.2153	2.9538	3.6922	1.4928	2.2391	2.9855	3.7319
.70	1.5109	2.2664	3.0218	3.7773	1.5256	2.2883	3.0511	3.8139
.72	1.5448	2.3171	3.0895	3.8619	1.5584	2.3375	3.1167	3.8959
.74	1.5790	2.3685	3.1581	3.9476	1.5913	2.3870	3.1827	3.9784
.76	1.6121	2.4182	3.2242	4.0303	1.6239	2.4359	3.2478	4.0598
.78	1.6455	2.4683	3.2911	4.1139	1.6558	2.4836	3.3115	4.1394
.80	1.6786	2.5178	3.3571	4.1964	1.6886	2.5329	3.3772	4.2215
.82	1.7112	2.5667	3.4223	4.2779	1.7197	2.5795	3.4394	4.2992
.84	1.7443	2.6164	3.4886	4.3607	1.7524	2.6286	3.5048	4.3810
.86	1.7764	2.6645	3.5527	4.4409	1.7828	2.6743	3.5657	4.4571
.88	1.8088	2.7132	3.6176	4.5220	1.8147	2.7221	3.6294	4.5368
.90	1.8408	2.7612	3.6815	4.6019	1.8454	2.7680	3.6907	4.6134
.92	1.8723	2.8085	3.7446	4.6808	1.8760	2.8140	3.7520	4.6900
.94	1.9053	2.8580	3.8106	4.7633	1.9082	2.8623	3.8164	4.7705
.96	1.9355	2.9033	3.8711	4.8389	1.9374	2.9061	3.8748	4.8436
.98	1.9670	2.9504	3.9339	4.9174	1.9679	2.9519	3.9358	4.9198
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7578	2.3452	2.8089	3.1813	1.7635	2.3582	2.8298	3.2104

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.50$

TABLE VI
(Continued)

$^f/y_o$	M = 15				M = 20			
	2Ic	3Ic	4Ic	5Ic	2Ic	3Ic	4Ic	5Ic
.40	1.0019	1.5028	2.0037	2.5046	1.0037	1.5055	2.0073	2.5092
.42	1.0394	1.5591	2.0788	2.5985	1.0411	1.5616	2.0821	2.6027
.44	1.0765	1.6147	2.1530	2.6912	1.0788	1.6173	2.1563	2.6954
.46	1.1133	1.6699	2.2266	2.7832	1.1150	1.6724	2.2299	2.7874
.48	1.1498	1.7246	2.2995	2.8744	1.1513	1.7270	2.3026	2.8782
.50	1.1855	1.7783	2.3710	2.9638	1.1870	1.7805	2.3740	2.9675
.52	1.2209	1.8314	2.4419	3.0523	1.2223	1.8335	2.4447	3.0559
.54	1.2562	1.8843	2.5124	3.1405	1.2575	1.8862	2.5149	3.1437
.56	1.2918	1.9377	2.5836	3.2296	1.2930	1.9395	2.5860	3.2325
.58	1.3255	1.9882	2.6509	3.3137	1.3267	1.9900	2.6534	3.3167
.60	1.3600	2.0400	2.7200	3.4000	1.3611	2.0416	2.7222	3.4027
.62	1.3936	2.0904	2.7873	3.4841	1.3949	2.0923	2.7898	3.4872
.64	1.4282	2.1422	2.8563	3.5704	1.4293	2.1439	2.8586	3.5732
.66	1.4619	2.1928	2.9238	3.6547	1.4628	2.1943	2.9257	3.6571
.68	1.4953	2.2430	2.9907	3.7383	1.4963	2.2445	2.9927	3.7408
.70	1.5282	2.2924	3.0565	3.8206	1.5293	2.2939	3.0586	3.8232
.72	1.5608	2.3412	3.1216	3.9020	1.5616	2.3425	3.1233	3.9041
.74	1.5935	2.3902	3.1870	3.9837	1.5941	2.3912	3.1883	3.9853
.76	1.6260	2.4390	3.2520	4.0650	1.6267	2.4400	3.2534	4.0667
.78	1.6578	2.4865	3.3154	4.1442	1.6582	2.4874	3.3165	4.1456
.80	1.6905	2.5357	3.3809	4.2262	1.6909	2.5364	3.3818	4.2273
.82	1.7213	2.5820	3.4426	4.3033	1.7218	2.5826	3.4435	4.3044
.84	1.7538	2.6307	3.5075	4.3844	1.7542	2.6313	3.5085	4.3856
.86	1.7840	2.6759	3.5679	4.4599	1.7844	2.6767	3.5689	4.4611
.88	1.8159	2.7238	3.6317	4.5397	1.8160	2.7241	3.6321	4.5401
.90	1.8464	2.7696	3.6928	4.6160	1.8466	2.7698	3.6931	4.6164
.92	1.8769	2.8153	3.7538	4.6922	1.8771	2.8156	3.7541	4.6926
.94	1.9088	2.8631	3.8175	4.7719	1.9089	2.8634	3.8179	4.7724
.96	1.9378	2.9067	3.8756	4.8445	1.9380	2.9070	3.8760	4.8450
.98	1.9681	2.9522	3.9362	4.9203	1.9681	2.9522	3.9362	4.9203
1.00	2.0000	3.0000	4.0000	5.0000	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat							
$.20y_o$	1.7645	2.3605	2.8333	3.2157	1.7648	2.3613	2.8347	3.2172

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, $\gamma = 1.50$

TABLE VI
(Continued)

f/y_o	M = 25			
	2Ic	3Ic	4Ic	5Ic
.40	1.0095	1.5142	2.0190	2.5237
.42	1.0419	1.5628	2.0838	2.6047
.44	1.0786	1.6179	2.1573	2.6966
.46	1.1157	1.6735	2.2314	2.7892
.48	1.1520	1.7280	2.3040	2.8800
.50	1.1877	1.7816	2.3754	2.9693
.52	1.2229	1.8344	2.4459	3.0574
.54	1.2580	1.8870	2.5160	3.1450
.56	1.2934	1.9401	2.5868	3.2335
.58	1.3271	1.9907	2.6543	3.3179
.60	1.3616	2.0423	2.7231	3.4039
.62	1.3954	2.0931	2.7908	3.4885
.64	1.4298	2.1447	2.8596	3.5745
.66	1.4633	2.1949	2.9265	3.6582
.68	1.4967	2.2450	2.9933	3.7417
.70	1.5296	2.2944	3.0593	3.8241
.72	1.5620	2.3430	3.1240	3.9050
.74	1.5945	2.3918	3.1890	3.9863
.76	1.6271	2.4406	3.2541	4.0677
.78	1.6586	2.4880	3.3173	4.1466
.80	1.6912	2.5368	3.3824	4.2280
.82	1.7219	2.5829	3.4439	4.3048
.84	1.7544	2.6316	3.5088	4.3860
.86	1.7846	2.6769	3.5692	4.4615
.88	1.8164	2.7245	3.6327	4.5409
.90	1.8467	2.7701	3.6934	4.6169
.92	1.8772	2.8158	3.7545	4.6931
.94	1.9089	2.8634	3.8179	4.7724
.96	1.9380	2.9070	3.8760	4.8450
.98	1.9683	2.9525	3.9366	4.9208
1.00	2.0000	3.0000	4.0000	5.0000

r	(d/r) ratio at nozzle throat			
$.20y_o$	1.7650	2.3617	2.8353	3.2178

Radii of circular loci in terms of r, on which magnetic pressure equals hydrodynamic pressure existing at nozzle y-coordinate ratio of (f/y_o), for various values of current and Mach Number, γ = 1.50

TABLE VII

CROSS-INTERPOLATION TABLE

f/y_o	$r=.20y_o$	$r=.25y_o$	$r=.33y_o$
1.00	1.00	1.00	1.00
.98	1.10	1.08	1.06
.96	1.20	1.16	1.12
.94	1.30	1.24	1.18
.92	1.40	1.32	1.24
.90	1.50	1.40	1.30
.88	1.60	1.48	1.36
.86	1.70	1.56	1.42
.84	1.80	1.64	1.48
.82	1.90	1.72	1.54
.80	2.00	1.80	1.60
.78	2.10	1.88	1.66
.76	2.20	1.96	1.72
.74	2.30	2.04	1.78
.72	2.40	2.12	1.84
.70	2.50	2.20	1.90
.68	2.60	2.28	1.96
.66	2.70	2.36	2.02
.64	2.80	2.44	2.08
.62	2.90	2.52	2.14
.60	3.00	2.60	2.20
.58	3.10	2.68	2.26
.56	3.20	2.76	2.32
.54	3.30	2.84	2.38
.52	3.40	2.92	2.44
.50	3.50	3.00	2.50
.48	3.60	3.08	2.56
.46	3.70	3.16	2.62
.44	3.80	3.24	2.68
.42	3.90	3.32	2.74
.40	4.00	3.40	2.80

Perpendicular distance in terms of r from current-carrying conductor to lines parallel to channel wall within the channel at ordinate f/y_o

BIBLIOGRAPHY

1. Alfvén, H.; Cosmical Electrodynamics; Clarendon Press, Oxford, 1950

2. Bateman, H.; Partial Differential Equations of Mathematical Physics; Dover Publications, New York, 1944

3. Bliss, Gilbert A.; Lectures on the Calculus of Variations; University of Chicago Press, Chicago, 1945

4. Braginskii, S. I.; Transport Phenomena in a Completely Ionized, Two-Temperature Plasma; Journal of Experimental and Theoretical Physics, (33) 358, February, 1958

5. Brand, Louis; Vector and Tensor Analysis; Wiley and Sons, New York, 1947

6. Clauser, Milton U.; Magnetohydrodynamics; Space Technology Laboratories Report PRL-8-04, June 1958

7. Courant, R.; Dirichlet's Principle, Conformal Mapping and Minimal Surfaces; Interscience, New York, 1950

8. Courant, Richard, Eugene Issacson and Mina Rees; On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences; Communications on Pure and Applied Mathematics, (5) 243, 1952

9. Forsyth, A. R.; Calculus of Variations; University Press, Cambridge, 1927

10. Fried, Burton D.; Flow through a Plane Magnetic Nozzle; Space Technology Laboratories Report ARL-6-40, November, 1956

11. Gunther, N. M.; La Théorie du Potentiel et ses Applications aux Problèmes Fondamentaux de la Physique Mathématique; Gauthiers-Villars, Paris, 1934

12. Jordan, Edward C.; Electromagnetic Waves and Radiating Systems; Prentice-Hall, New York, 1950

13. Lamb, Sir Horace; Hydrodynamics; University Press, Cambridge, 1930

14. Landshoff, Rolf K. M.; Magnetohydrodynamics; Stanford University Press, Palo Alto, 1957

15. Liepman, Hans W., and Allen E. Puckett; Aerodynamics of a Compressible Fluid; Wiley and Sons, New York, 1947

16. Liepman, Hans W., and A. Roshko; Elements of Gas Dynamics; Wiley and Sons, New York, 1957

BIBLIOGRAPHY
continued

17. Loughhead, R. E.; Solution of Problems Involving the Hydromagnetic Flow of Compressible Ionized Fluids; Physical Review, (99) 1678, July-September, 1955

18. Lovitt, William V.; Linear Integral Equations; McGraw-Hill, New York, 1924

19. Meyer, Rudolph C.; On Reducing Aerodynamic Heat Transfer Rates by Magnetohydrodynamic Techniques; Institute of the Aeronautical Sciences Preprint No. 816, January, 1958

20. Mikhlin, S. G.; Integral Equations; Pergamon Press, New York, 1957

21. Milne-Thompson, L. M.; Theoretical Hydrodynamics; McMillan, London, 1938

22. Murnaghan, Francis D.; Introduction to Applied Mathematics; Wiley and Sons, New York, 1948

23. Paley, R. E. A. C., and Norbert Weiner; Fourier Transforms, American Mathematical Society Colloquium Publications, Volume XIX; Waverly Press, Baltimore, 1934

24. Patrick, Richard M.; Magnetohydrodynamics of Compressible Fluids; Doctoral Dissertation, Cornell University, 1956

25. Sneddon, Ian M.; Fourier Transforms; McGraw-Hill, New York, 1951

26. Synge, J. L.; The Hypercircle in Mathematical Physics; University Press, Cambridge, 1957

27. Tayler, R. J.; The Influence of an Axial Magnetic Field on the Stability of a Constricted Gas Discharge; Proceedings of the Physical Society, (70) 1049, November 1957

28. Tonks, Lewi; Particle Transport, Electric Currents, and Pressure Balance in a Magnetically Immobilized Plasma; Physical Review, (97) 1443, January-March 1955

29. Webster, Arthur G; Partial Differential Equations of Mathematical Physics; Hafner, New York, 1947

30. Wiley, C. R., Jr.; Advanced Engineering Mathematics; McGraw-Hill, New York, 1951

thesG865
Magnetic nozzling of a plasma flow.
3 2768 001 03694 0
DUDLEY KNOX LIBRARY